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Phase space analysis of quintessence fields trapped in a Randall-Sundrum Braneworld: anisotropic Bianchi I brane with a negative dark radiation term

Dagoberto Escobar,[1, *] Carlos R. Fadragas,[2, †] Genly Leon,[3, ‡] and Yoelsy Leyva[4, §]

[1]*Departamento de Física Universidad de Camagüey, Cuba.*
[2]*Departamento de Física, Universidad Central de Las Villas, 54830 Santa Clara, Cuba.*
[3]*Instituto de Física, Pontificia Universidad Católica de Valparaíso, Casilla 4950, Valparaíso, Chile*
[4]*División de Ciencias e Ingeniería de la Universidad de Guanajuato, A.P. 150, 37150, León, Guanajuato, México.*

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In this work we present a phase space analysis of a quintessence field and a perfect fluid trapped in a Randall-Sundrum's Braneworld of type 2. We consider a homogeneous but anisotropic Bianchi I (BI) brane geometry. Moreover, we consider the effect of the projection of the five-dimensional Weyl tensor onto the three-brane in the form of a negative Dark Radiation term. For the treatment of the potential we use the "Method of f-devisers" that allow investigating arbitrary potentials in a phase space. We present general conditions on the potential in order to obtain the stability of standard 4D and non-standard 5D de Sitter solutions, and we provide the stability conditions for both scalar field-matter scaling solutions, scalar field-dark radiation solutions and scalar field-dominated solutions. We find that the shear-dominated solutions are unstable (particularly, contracting shear-dominated solutions are of saddle type). As a main difference with our previous work, the traditionally ever-expanding models could potentially re-collapse due to the negativity of the dark radiation. Additionally, our system admits a large class of static solutions that are of saddle type. This kind of solutions are important at intermediate stages in the evolution of the universe, since they allow to the transition from contracting to expanding models and vice versa. New features of our scenario are the existence of a bounce and a turnaround, which leads to cyclic behavior, that are not allowed in Bianchi I branes with positive dark radiation term. Finally, as specific examples we consider the potentials $V \propto \sinh^{-\alpha}(\beta\phi)$ and $V \propto [\cosh(\xi\phi) - 1]$ which have simple f-devisers.

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I. INTRODUCTION

The idea that our Universe is a brane embedded in a 5-dimensional space, on which the particles of the standard model are confined, while gravity is allowed to propagate also in the bulk of the higher-dimensional manifold was proposed in [1, 2]. This braneworlds models were baptized as a Randall-Sundrum type 1 (RS1) and 2 (RS2) [1]. One of them, the RS2 model, found a great importance in cosmology and has been studied from different angles, because of the cosmological field equations on the brane have modification [3–6] that have significant impact on inflationary scenario [7–9].

The phase space of the Randall-Sundrum braneworld has been investigated widely in the literature [10, 11, 13–17]. In [18, 19] the asymptotic behavior of FRLW, Bianchi I and V brane in the presence of perfect fluid was studied [2] taking into account the contribution of the non-vanishing dark radiation term $\mathcal{U}$. Goheer and Dunsby [20, 21] used the dynamical system methods to the analysis of how the FLRW and Bianchi I scalar field models with and exponential potential is affected by the bulk Weyl tensor. These studies was extended to arbitrary potentials in FRW branes [22, 23] while the contribution of the positive dark radiation term for arbitrary potentials in Bianchi I branes was generalized in [24].

The dark radiation term $\mathcal{U}$ represents the scalar component of the electric (Coulomb) part of the five-dimensional Weyl tensor of the bulk [25, 26]. $\mathcal{U}$ scales just like radiation with a constant μ, i.e., $\mathcal{U} = \mu a^{-4}$. For that reason it is called

*Electronic address: dagoberto.escobar@reduc.edu.cu
†Electronic address: fadragas@uclv.edu.cu
‡Electronic address: genly.leon@ucv.cl
§Electronic address: yoelsy.leyva@fisica.ugto.mx

[1] The original motivations were to looking for an explanation to the hierarchy problem [2], in (RS1), and an alternative mechanism to the Kaluza-Klein compactifications [1], in (RS2).
[2] A complementary analysis for FLRW branes was carry on in [10] using a 2-dimensional Hamiltonian system.

the dark radiation. The coefficient μ is a constant of integration obtained by integrating the five-dimensional Einstein equations [3, 4, 27–32]. In FRW brane model, μ is related with the mass of the black hole in the bulk therefore $\mathcal{U}$ is positive defined [33–35]. For anisotropic models both positive and negative μ are possible mathematically [19, 21, 36]. Its magnitude and sign can depend on the choice of initial conditions when solving the five-dimensional Einstein equation. Hence, even the sign of μ remains an open question [26, 34].

Dark radiation should strongly affect both big-bang nucleosynthesis (BBN) and the cosmic microwave background (CMB). Hence, such observations can be used to constrain both the magnitude and sign of the dark radiation as well as any source of radiation in the model (see the classical references [37, 38], see the appendix of [39] and [40–43]). In [43], the constraints on BBN alone allow for $-1.23 \leq \rho_{DR}/\rho_\gamma \leq 0.11$ in a dark radiation component, where ρ_γ is the total energy density in background photons just before the BBN epoch at $T = 1MeV$, $\rho_{DR} = 6\mathcal{U}/\lambda$, where λ is the tension of the brane. In order to compute the theoretical prediction of CMB anisotropies exactly, one must eventually solve the perturbations including the contribution from the bulk. However, in [43] the CMB power spectrum was used to constrain the dominant expansion-rate effect of the dark-radiation term in the generalized Friedmann equation. If the constraint from effects of dark radiation on the expansion rate are included, the allowed concordance range range of dark radiation content reduces to $-0.41 \leq \rho_{DR}/\rho_\gamma \leq 0.105$ at the 95% confidence level. So, in principle, the data allow for a negative extra radiation. Considering negative values of dark radiation in Bianchi I models leads to very interesting solutions for which the universe can both collapse or isotropize [18–21, 44]. All the above reasons motivate the investigation of a homogeneous but anisotropic brane model with $\mathcal{U} < 0$.

In this paper, we study the dynamics of an scalar field with arbitrary potential trapped in a Randall-Sundrum's Braneworld of type 2. We consider a homogeneous but anisotropic Bianchi I (BI) brane filled also with a perfect fluid. Furthermore, we consider the effect of the projection of the five-dimensional Weyl tensor onto the three-brane in the form of a negative Dark Radiation term. The results presented here complement our previous investigation in [24] where we considered the effect of a positive Dark Radiation term on the brane.

For the treatment of the potential we use a modification of the method that has been applied to isotropic (FRW) scenarios [22, 45–49], and that has been generalized to several cosmological contexts [23, 24, 50, 51]. The modified method, that we call "Method of f-devisers", allows us to perform a phase-space analysis of a cosmological model, without the need for specifying the potential. This is a significant advantage, since one can first perform the analysis for arbitrary potentials and then just substitute the desired forms, instead of repeating the whole procedure for every distinct potential. This investigation represents a further step in a series of works devoted to the use of the general procedure of f-devisers for investigating both FRW and Bianchi I branes, initiated in our previous works [23, 24].

A main difference with our results in [24], is that the traditionally ever-expanding models could potentially recollapse due to the negativity of the dark radiation ($\mathcal{U} < 0$). Also, our system admits a large class of static solutions that are of saddle type. This kind of solutions are important at intermediated stages in the evolution of the universe since they allow for the transition from expanding to contracting models, and vice versa. New features of our scenario are the possibility of a bounce and a turnaround, which leads to cyclic behavior. This behavior is not allowed in Bianchi I branes with positive dark radiation term [24].

The interest of static solutions in the cosmological setting goes back to the discovering of Einstein static (ES) model. This model was proposed by Einstein as an attempt to incorporate Mach's principle (geometry or motion do not make sense in an empty universe) into the general relativity (GR) and also to overcome the boundary conditions of the theory [52]. The issue of the stability of ES model has been studied several times after the classical paper by Eddington [53] where it was showed the instability of ES solution with respect to the homogeneous and isotropic perturbations. This fact is the key feature that allows for the transition between the Friedmann transient matter-dominated deceleration phase, to the current accelerated expansion. In [54], and [55], were considered inhomogeneous and anisotropic perturbations to ES model. A cosmology including a mixture of isotropic radiation and a ghost massless scalar field admits an almost static solution that it is stable against homogeneous an isotropic perturbations. These solutions are also stable against the three types of linear inhomogeneous perturbations: density, vorticity and gravitational-wave perturbations. However, ES model becomes unstable (it no longer exists) if large-amplitude homogeneous and anisotropic perturbations are introduced [56]. The stability issue of ES models has acquired renewed interest, for example, in the context of $f(R)$ theories due the success of such theories in explaining the cosmic acceleration and because ES solutions can provide the link between decelerating/accelerating phases in these theories in the same way as in GR [57]. The stability analysis in [57] shows that as in GR [58], the Einstein static model is neutrally stable against vector and tensor perturbations for all equations of state at all scales. In [59], is presented a complete picture of how ES models fits into the overall phase space structure of $f(R)$ gravity, complementing the perturbation analysis made in their previous work [57]. There the authors investigate the prototype of $f(R)$-theory compatible with an static solution obtained in [57]. Several unexpected cosmological features were obtained such as the existence of no big-bang scenario for any realistic equation of state, but only bounce scenarios or expansion after an initial Minkowski phase; the existence of past and a future bounce, a phenomenon which is absent in general relativity; the unfolding of the quasi-periodic nature of ES model, in the sense that orbits emerging from a source,

spiral around ES solution for some time and goes to a sink for a negative equation of state parameter of the perfect fluid. The stability analysis of the ES universe, and other kinds of static solutions in the context of brane cosmology with dust matter and anisotropic geometry was presented in [18, 19]. Here we present several classes of static solution for a Bianchi I brane containing an scalar field with arbitrary potential. This class of solutions are more general than the presented in in [18, 19]. As commented before, our static solutions and very important since they allow to the transition from a contracting to an expanding universe, and vice versa.

The paper is organized as follows. In section II we present our general approach for the investigation of arbitrary potentials. In section III are presented the cosmological equations of our model. In section IV we proceed to the dynamical system analysis of the cosmological model under consideration. For this purpose we use the method of f- devisers discussed in section II, and we introduce local charts adapted to each of the more interesting singular points. In this way, we analyze the stability of the shear-dominated solutions, the stability of the de Sitter solutions and the stability of the solutions with 5D corrections. In section V we summarize our analytical results and proceed to the physical discussion specially to the comparison with previous results. Particular emphasis is made on static solutions. Our analytical results are illustrated in section VI for the potentials $V(\phi) = V_0 \left[\cosh(\xi\phi) - 1\right]$ (see references [22, 46, 48, 60–66]) and $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ (see references [22, 48, 61, 63–65, 67]) which have simple f-devisers. Is worthy to mention that this procedure is general and applies to other potentials different from the cosh-like, the inverted sinh-like and the exponential one. Finally, in section VII are drawn our general conclusions.

II. "METHOD OF f-DEVISERS"

In this section we present a general method that allows us to perform a phase-space analysis of a cosmological model, without the need for specifying the potential. This procedure that we will call "Method of f-devisers" consist of a modification of a method used in isotropic (FRW) scalar field cosmologies for the treatment of the potential of the scalar field [22, 45–48], that has been generalized to several cosmological contexts [23, 24, 50, 51]. An earlier attempt to introduce the "Method of f-devisers" as in the present form was in [46] in the context of FRW cosmologies. However, there the authors restricted their attention to a scalar field dark-matter model with a cosh-like potential, and there was revealed that the late-time attractor is always the de Sitter solution. In section VI A we will take this potential as an application example of the present "Method of f-devisers" in the context of brane cosmology. The method as introduced here applies to a more general class of potentials than the cosh-like. In fact, the method has the significant advantage that one can first perform the analysis for arbitrary potentials and then just substitute the desired forms, instead of repeating the whole procedure for every given potential. More importantly is that the method does not depend on the cosmological scenario, since it is constructed on the scalar field and its self-interacting potential, and can be generalized to several scalar fields (for example to quintom models [68]). The disadvantages are that such a generalization is only possible if in the model do not appear functions that contains mixed terms of several of the scalar fields. For example, cosmological models containing an interaction term given by $V(\phi, \varphi)$ discussed in references [69] and [70], where ϕ, φ are the scalar fields, cannot be investigated using the method of f-devisers. On the other hand, in the single field case the method cannot be applied if the model contains more than one "arbitrary" function depending on the scalar field, as for example a cosmological models containing a potential $V(\phi)$ and coupling function $\chi(\phi)$, that appeared in the context of non-minimally coupled scalar field cosmologies [71–73]. In such a case the more convenient approach is to consider the scalar field itself as a dynamical variable.

The procedure for deriving the "Method of f-devisers" is as follows. Let us define the two dynamical variables

$$s = -\frac{V'(\phi)}{V(\phi)}, \tag{1}$$

$$f = \frac{V''(\phi)}{V(\phi)} - \frac{V'(\phi)^2}{V(\phi)^2}, \tag{2}$$

while keeping the potential still arbitrary.

In the formulas (1) and (2), the primes denote differentiation with respect to ϕ. Now, a simple observation is that for general physical potentials, and for the usual ansatzes of the cosmological literature, it results that f can be expressed as an explicit one-valued function of s, that is $f = f(s)$ (as can be seen in Table I), and therefore we result to a closed dynamical system for s and a set of normalized-variables.

Conversely, it is possible to reconstruct the corresponding potential from a given $f(s)$. In particular, one starts

TABLE I: Explicit for the function $f(s)$ for some quintessence potentials.

Potential	References	$f(s)$
$V(\phi)=V_0\left[\cosh\left(\xi\phi\right)-1\right]$	[22, 46, 48, 60–66]	$-\frac{1}{2}(s^2-\xi^2)$
$V(\phi)=V_0\sinh^{-\alpha}(\beta\phi)$	[22, 48, 61, 63–65, 67]	$\frac{s^2}{\alpha}-\alpha\beta^2$
$V(\phi)=V_0e^{-\lambda\phi}+\Lambda$	[63, 74]	$-s(s-\lambda)$
$V(\phi)=V_0\left[e^{\alpha\phi}+e^{\beta\phi}\right]$	[75–77]	$-(s+\alpha)(s+\beta)$

with

$$\frac{ds}{d\phi}=-f(s), \tag{3}$$

$$\frac{dV}{d\phi}=-sV, \tag{4}$$

which give

$$\phi(s) = \phi_0-\int_{s_0}^{s}\frac{1}{f(K)}\,dK \tag{5}$$

$$V(s) = e^{\int_{s_0}^{s}\frac{K}{f(K)}\,dK}\bar{V}_0, \tag{6}$$

where the integration constants satisfies $V(s_0)=\bar{V}_0$, $\phi(s_0)=\phi_0$. We mention that relations (5) and (6) are always valid and they provide the potential in an implicit form. However, for the usual cosmological cases of Table I the potential can be written explicitly, that is $V=V(\phi)$, after elimination of s between (5) and (6).

The direct derivation of the function $f(s)$ from a given $V(\phi)$, as well as the reconstruction procedure for obtaining $V(\phi)$ from a given f-function is what we call "Method of f-devisers". In the section IV we use this method for investigating the phase portrait of quintessence fields trapped in a Randall-Sundrum Braneworld in the case of an anisotropic Bianchi I brane with a negative dark radiation term.

III. BIANCHI COSMOLOGY ON THE BRANE AND QUALITATIVE BEHAVIOR

Having presented the method we will use in the present paper, now let us present the cosmological model. We will be concerned here with the dynamics of a self interacting scalar field with an arbitrary self-interaction potential, that is trapped in a Randall-Sundrum brane of type 2 (RS2). The field equations for the Bianchi I metric are the following: [3]

$$H^2=\frac{1}{3}\rho_T\left(1+\frac{\rho_T}{2\lambda}\right)+\frac{1}{3}\sigma^2+\frac{2\mathcal{U}}{\lambda} \tag{7}$$

$$\dot{H}=-\frac{1}{2}\left(1+\frac{\rho_T}{\lambda}\right)\left(\dot{\phi}^2+\gamma\rho_m\right)-\frac{4\mathcal{U}}{\lambda}-\sigma^2 \tag{8}$$

$$\dot{\sigma}=-3H\sigma \tag{9}$$

$$\dot{\rho}_m+3H(\rho_m+p_m)=0 \tag{10}$$

[3] See our previous paper [24] and and therein references for a full derivations of the effective Einstein equations for Bianchi I models.

$$\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0, \tag{11}$$

where we have used units in which $\kappa^2 = 8\phi G = 1$. In the above expressions H denotes the Hubble factor, σ is the shear term, while

$$\rho_T = \rho_\phi + \rho_m \tag{12}$$

is the total energy density content of the universe, ρ_m corresponds to the energy density of a barotropic fluid with equation of state

$$p_m = (\gamma - 1)\rho_m, \tag{13}$$

and

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{14}$$

is the energy density of the scalar field ϕ. $\mathcal{U}$ denotes the so-called Dark Radiation term which in this paper is assumed to be negative, $\mathcal{U} < 0$, as it could be the case according to observations [37, 38]. From the equations (7) and (8), and from the fact that $\mathcal{U} < 0$, it follows that H can become zero and moreover change its sign during the evolution. Thus, for $\mathcal{U} < 0$ the expanding phase could experience the cosmological turnaround triggered by the negativity of this dark radiation, and similarly the contracting phase can be led to a cosmological bounce and then to expansion.

Before proceeding to the detailed investigation of the model (7)-(11) using dynamical systems tools, let us discuss the above system using a heuristic reasoning. Integrating equation (9) and (10) we obtain that $\sigma = \sigma_0 a^{-3}$ and $\rho_m = \rho_{m0} a^{-3\gamma}$, while as discussed in the introduction $\mathcal{U} = \mu a^{-4}$. Let us examine the case of $\frac{\dot{\phi}^2}{2} \gg V(\phi)$, thus, $\rho_\phi \approx \rho_{\phi 0} a^{-6}$, which implies $\rho_T = \rho_{\phi 0} a^{-6} + \rho_{m0} a^{-3\gamma}$. Then equation (7) reduces to

$$3H^2 = \frac{\rho_{m0}^2 a^{-6\gamma}}{2\lambda} + \frac{\rho_{m0}\rho_{\phi 0} a^{-3\gamma-6}}{\lambda} + \frac{\rho_{\phi 0}^2}{2\lambda} a^{-12} + \left(\rho_{\phi 0} + \sigma_0^2\right) a^{-6} + \rho_{m0} a^{-3\gamma} - \frac{6|\mu|}{\lambda} a^{-4} \tag{15}$$

There are several cases depending of the values of γ. Suppose that the universe lies currently in the usual expanding phase, i.e. $H(a = a_0) > 0$, where a_0 denotes the scale of the universe at the present time. We take as a reference universe state the present expanding one. Thus a future zero of H will correspond to a turnaround, while a past one corresponds to a bounce.

- For $\gamma > 4/3$ the dark radiation term in equation (15) falls less rapidly than the matter terms, making the future turnaround inevitable for any initial negative $\mathcal{U}$.

- For $\gamma = 4/3$, the last two terms in the r.h.s. of (15) are equally important in a low-energy regime, and the condition for the future turnaround is $\rho_{m0} < 6|\mu|/\lambda$.

- For $1 < \gamma < \frac{4}{3}$, the leading terms in the r.h.s. of (15) that are equally important are the last two terms (since the terms scaling as a^{-6} goes very fast to zero). As a result, we obtain that the condition $H = 0$ leads to the equation $\rho_{m0} a^{4-3\gamma} - \frac{6|\mu|}{\lambda} = 0$. This solution as a positive root given by $a_+ = \left[\frac{6|\mu|}{\lambda \rho_{m0}}\right]^{\frac{1}{4-3\gamma}}$. If $a_+ > a_0$ (the scale factor today) it corresponds to the future of the Universe under consideration and indicates the point of turnaround. If $a_+ < a_0$, will describe the past and corresponds to a bounce.

- For $\gamma = 1$, the leading terms in the r.h.s. of (15) that are equally important are the first and the last three terms. As a result, we obtain that the condition $H = 0$ leads to the cubic equation for the scale factor $a^3 - \frac{6|\mu|}{\lambda \rho_{m0}} a^2 + \left[\frac{\rho_{m0}}{2\lambda} + \frac{\rho_{\phi 0}}{\rho_{m0}} + \frac{\sigma_0^2}{\rho_{m0}}\right] = 0$. Applying Descartes's rule, this equation has always one negative root and two real positive or either complex conjugated roots. The condition for the existence of two positive roots is

$$\rho_{m0}^4 + 2\lambda \rho_{m0}^3 (\rho_{\phi 0} + \sigma_0^2) < \frac{64|\mu|^3}{\lambda^2}.$$

  This expression reduces to the analogous expression in [44] for the case of a perfect fluid under the assumption $\rho_{\phi 0} = \sigma_0 = 0$. A root which is bigger than a_0 (the scale factor today) corresponds to the future of the Universe under consideration and indicates the point of turnaround. A root which is less than a_0, will describe the past and corresponds to a bounce.

- For $\frac{2}{3} < \gamma < 1$, the leading terms in the r.h.s. of (15) that are equally important are the first and the last two terms. As a result, we obtain that the condition $H = 0$ leads to the equation
$$-\frac{6\mu a^{6\gamma-4}}{\lambda} + \rho_{m0} a^{3\gamma} + \frac{\rho_{m0}^2}{2\lambda} = 0.$$
This equation must be solved numerically looking for real positive roots. A root which is bigger than a_0 (the scale factor today) corresponds to the future of the Universe under consideration and indicates the point of turnaround. A root which is less than a_0, will describe the past and corresponds to a bounce.

- For $\gamma = \frac{2}{3}$, the leading terms in the r.h.s. of (15) that are equally important are the first and the last two terms. As a result, we obtain that the condition $H = 0$ leads to the quadratic equation for the scale factor $a^2 + \left(\frac{\rho_{m0}}{2\lambda} - \frac{6|\mu|}{\lambda\rho_{m0}}\right) = 0$. The condition for the existence of a positive root, $a_+ = \sqrt{\left(\frac{\rho_{m0}}{2\lambda} - \frac{6|\mu|}{\lambda\rho_{m0}}\right)}$, is $\rho_{m0}^2 < 12|\mu|$. If $a_+ > a_0$ (the scale factor today) corresponds to the future of the Universe under consideration and indicates the point of turnaround. If $a_+ < a_0$, it describes the past and corresponds to a bounce.

- For $\gamma < 2/3$, even the term $\propto a^{-6\gamma}$, falls less rapidly than $\mathcal{U}$, and the turnaround in the future becomes impossible.

Let us examine the case of $\frac{\dot{\phi}^2}{2} \ll V(\phi)$, thus, $\rho_\phi \approx V(\phi(a))$. Then, $\rho_T = V(\phi(a)) + \rho_{m0} a^{-3\gamma}$. Then equation (7) reduces to
$$3H^2 = V\left[1 + \frac{V}{2\lambda}\right] + \frac{\rho_{m0}^2 a^{-6\gamma}}{2\lambda} + \frac{\sigma_0^2}{a^6} + \rho_{m0}\left[1 + \frac{V}{\lambda}\right] a^{-3\gamma} - \frac{6|\mu|}{a^4\lambda}. \tag{16}$$
Let consider the simpler case of a potential that tends asymptotically to a positive constant $V = V_0 > 0$.

- For $\gamma > \frac{4}{3}$, the leading terms in the r.h.s. of (16) are the first and the last one. The condition $H = 0$ leads to the quartic equation for the scale factor
$$V_0\left[1 + \frac{V_0}{2\lambda}\right] a^4 - \frac{6|\mu|}{\lambda} = 0.$$
This equation has one negative solution, two purely imaginary solutions, and a positive one given by $a_+ = \frac{\sqrt{2}\sqrt[4]{3}\sqrt[4]{\mu}}{\sqrt[4]{V_0(2\lambda+V_0)}}$. If $a_+ > a_0$ (the scale factor today), it corresponds to the future of the Universe under consideration and indicates the point of turnaround. If $a_+ < a_0$, it describes the past and corresponds to a bounce.

- For $\gamma = \frac{4}{3}$, the leading terms in the r.h.s. of (16) are the first and the last two terms. The condition $H = 0$ leads to the quartic equation for the scale factor
$$V_0\left[1 + \frac{V_0}{2\lambda}\right] a^4 + \rho_{m0}\left[1 + \frac{V_0}{\lambda}\right] - \frac{6|\mu|}{\lambda} = 0.$$
The condition for the future collapse is $\rho_{m0}\left[1 + \frac{V_0}{\lambda}\right] < \frac{6|\mu|}{\lambda}$.

- For $1 < \gamma < \frac{4}{3}$, the leading terms in the r.h.s. of (16) are the first and the last two terms. The condition $H = 0$ leads to the equation for the scale factor
$$V_0\left[1 + \frac{V_0}{2\lambda}\right] a^4 + \rho_{m0}\left[1 + \frac{V_0}{\lambda}\right] a^{4-3\gamma} - \frac{6|\mu|}{\lambda} = 0.$$
This equation must be solved numerically looking for real positive roots. A root which is bigger than a_0 (the scale factor today) corresponds to the future of the Universe under consideration and indicates the point of turnaround. A root which is less than a_0, will describe the past and corresponds to a bounce.

- For $\gamma = 1$, the leading terms in the r.h.s. of (16) are the first and the last two terms. The condition $H = 0$ leads to the quartic equation for the scale factor
$$V_0\left[1 + \frac{V_0}{2\lambda}\right] a^4 + \rho_{m0}\left[1 + \frac{V_0}{\lambda}\right] a - \frac{6|\mu|}{\lambda} = 0.$$
Applying Descartes's rule this equation has one positive root, one negative root and two complex conjugated roots. If the positive root is bigger than a_0 (the scale factor today), it corresponds to the future of the Universe under consideration and indicates the point of turnaround. If it is less than a_0, it describes the past and corresponds to a bounce.

- For $\frac{2}{3} < \gamma < 1$, all the terms in the r.h.s. of (16) but the third, are relevant. The condition $H = 0$ leads to the equation for the scale factor

$$-\frac{6\mu a^{6\gamma-4}}{\lambda} + \frac{\rho_{m0} a^{3\gamma}(\lambda + V_0)}{\lambda} + \frac{V_0 a^{6\gamma}(2\lambda + V_0)}{2\lambda} + \frac{\rho_{m0}^2}{2\lambda} = 0.$$

  This equation must be solved numerically looking for real positive roots. A root which is bigger than a_0 (the scale factor today) corresponds to the future of the Universe under consideration and indicates the point of turnaround. A root which is less than a_0, will describe the past and corresponds to a bounce.

- For $\gamma = \frac{2}{3}$, all the terms in the r.h.s. of (16) but the third, are relevant. The condition $H = 0$ leads to the quartic equation for the scale factor

$$V_0 \left[1 + \frac{V_0}{2\lambda}\right] a^4 + \rho_{m0} \left[1 + \frac{V_0}{\lambda}\right] a^2 + \frac{\rho_{m0}^2}{2\lambda} - \frac{6|\mu|}{\lambda} = 0.$$

  For $\mu > \frac{\rho_{m0}^2}{12}$ there is one positive solution given by $a_+ = \sqrt{\frac{\sqrt{\lambda^2\rho_{m0}^2 + 12\mu V_0(2\lambda+V_0)} + \rho_{m0}(-\lambda-V_0)}{V_0(2\lambda+V_0)}}$, one negative solution $a_- = -\sqrt{\frac{\sqrt{\lambda^2\rho_{m0}^2 + 12\mu V_0(2\lambda+V_0)} + \rho_{m0}(-\lambda-V_0)}{V_0(2\lambda+V_0)}}$, and two complex conjugated. Otherwise the four are complex conjugated. Thus, the condition for the bounce or the collapse is $\mu > \frac{\rho_{m0}^2}{12}$. If the positive root is bigger than a_0 (the scale factor today), it corresponds to the future of the Universe under consideration and indicates the point of turnaround. If it is less than a_0, it describes the past and corresponds to a bounce.

- For $\gamma < 2/3$, even the term $\propto a^{-6\gamma}$, falls less rapidly than $\mathcal{U}$, and the turnaround in the future becomes impossible.

Summarizing, using a heuristic reasoning we have obtained several conditions for the existence of bounces and turnaround of cosmological solutions for a massless scalar field and a scalar field with a potential asymptotically constant. For the analysis of more general potentials we submit the reader to the reference [11] where the authors discuss conditions for bounces and they provide details of chaotic behavior in the isotropic brane model containing an scalar field for a negative dark radiation $\mathcal{U} < 0$.

IV. DETAILED DYNAMICAL ANALYSIS FOR BIANCHI I BRANE WITH NEGATIVE DARK RADIATION TERM

After the qualitative analysis of the previous section, in the present section we investigate the cosmological model (7)-(11) from the dynamical systems perspective. First we need to recast the cosmological equations (7)-(11) as an autonomous system of first order ordinary differential equations [72, 73, 78–80]. In order to avoid ambiguities related to the non-compactness at infinity we define compact variables that can describe both expanding and collapsing models [16, 18, 81–86]:

$$\begin{aligned} Q = \frac{H}{D}, x = \frac{\dot{\phi}}{\sqrt{6}D}, y = \frac{V}{3H^2}, \Omega_\lambda = \frac{\rho_T^2}{6\lambda D^2}, \\ \Omega_m = \frac{\rho_m}{3D^2}, \Sigma = \frac{\sigma}{\sqrt{3}D}, \Omega_U = -\frac{2\mathcal{U}}{\lambda D^2}, \end{aligned} \tag{17}$$

where $D = \sqrt{H^2 - \frac{2U}{\lambda}}$;

For the scalar potential treatment, we proceed following the method introduced in section II.

Using the Friedmann equation (8) we obtain the following relation between the variables (17)

$$x^2 + y + \Omega_m + \Omega_\lambda + \Sigma^2 = 1 \tag{18}$$

The restriction (18) allows to forget about one of the dynamical variables, e.g., y, to obtaining a reduced dynamical system. From the condition $0 \le y \le 1$ we have the following inequality

$$0 \le x^2 + \Omega_m + \Omega_\lambda + \Sigma^2 \le 1 \tag{19}$$

The definition of D leads to the additional restriction

$$Q^2 + \Omega_U = 1. \tag{20}$$

Thus, restriction (20) allows to eliminate another degree of freedom, namely variable Ω_U.

Using the variables (17), the field equations (7)-(11) and the new time variable $d\tau = Ddt$, we obtain the following autonomous system of ordinary differential equations (ODE):

$$Q' = \frac{1}{2}\left(Q^2-1\right)\left(3\gamma\Omega_m + 6\Sigma^2 + 6x^2 - 4\right) + \frac{3}{2}\Xi\left(Q^2-1\right) \tag{21}$$

$$x' = \frac{1}{2}\left(3Qx\left(\gamma\Omega_m + 2\Sigma^2 + 2x^2 - 2\right) + \sqrt{6}s\left(1 - x^2 - \Omega_m - \Omega_\lambda - \Sigma^2\right)\right) + \frac{3}{2}\Xi Q x \tag{22}$$

$$\Omega_m' = 3Q\Omega_m\left(\gamma(\Omega_m - 1) + 2\Sigma^2 + 2x^2\right) + 3\Xi Q\Omega_m \tag{23}$$

$$\Omega_\lambda' = 3Q\Omega_\lambda\left(\gamma\Omega_m + 2\Sigma^2 + 2x^2\right) + 3\Xi Q(\Omega_\lambda - 1) \tag{24}$$

$$\Sigma' = \frac{3}{2}Q\Sigma\left(\gamma\Omega_m + 2\Sigma^2 + 2x^2 - 2\right) + \frac{3}{2}\Xi Q\Sigma \tag{25}$$

$$s' = -\sqrt{6}xf(s). \tag{26}$$

Where the comma denotes derivatives with respect to τ, and

$$\Xi \equiv \frac{\rho_T}{\lambda}\left(\gamma\Omega_m + 2x^2\right), \tag{27}$$

From (18) follows the useful relationship

$$\frac{\rho_T}{\lambda} = \frac{2\Omega_\lambda}{x^2 + y + \Omega_m} = \frac{2\Omega_\lambda}{1 - \Omega_\lambda - \Sigma^2}. \tag{28}$$

It is easy to see from (28) and the definition (27) that

$$\Xi = \frac{2\Omega_\lambda\left(\gamma\Omega_m + 2x^2\right)}{1 - \Omega_\lambda - \Sigma^2} \tag{29}$$

From (28) follows that the region $1-\Omega_\lambda-\Sigma^2 \equiv \Omega_m+x^2+y = 0$ corresponds to cosmological solutions where $\rho_T \gg \lambda$ (corresponding to the formal limit $\lambda \to 0$). Therefore, they are associate to high energy regions, i.e., to cosmological solutions in a neighborhood of the initial singularity [4]. Due to its classic nature, our model is not appropriate to describing the dynamics near the initial singularity, where quantum effects appear. However, from the mathematical viewpoint, these region ($\Omega_\lambda + \Sigma^2 = 1$) is reached asymptotically. In fact, as some numerical integrations corroborate, there exists an open set of orbits in the phase interior that tends to the boundary $\Omega_\lambda+\Sigma^2 = 1$ as $\tau \to -\infty$. Therefore, for mathematical motivations it is common to attach the boundary $\Omega_\lambda + \Sigma^2 = 1$ to the phase space [5]. On the other hand the points with ($\Omega_\lambda = 0$) are associated to the standard 4D behavior ($\rho_T \ll \lambda$ or $\lambda \to \infty$) and corresponds to the low energy regime.

From definition (17) and from the restriction (18), and taking into account the previous statements, it is enough to investigate to the flow of (22)-(26) defined in the phase space

$$\begin{aligned}\Psi = \{(Q, x, \Omega_m, \Omega_\lambda, \Sigma) : 0 \leq x^2 + \Omega_m + \Omega_\lambda + \Sigma^2 \leq 1, -1 \leq Q \leq 1,\\ -1 \leq x \leq 1, 0 \leq \Omega_m \leq 1, 0 \leq \Omega_\lambda \leq 1, -1 \leq \Sigma \leq 1\} \times \{s \in \mathbb{R}\}.\end{aligned} \tag{30}$$

The system (22)-(26) admits twenty four classes of (curves of) fixed points associated to expanding (contracting solutions). Its coordinates in the phase space are given in Table II. Note that for $U = 0$ are recovered the models that we have been investigated previously in [23]. As we commented previously, our model is not applicable near initial singularity (that is at the singular surface $1 - \Omega_\lambda - \Sigma^2 = 0$). However, in principle we can apply a similar approach as in [24] to investigate the dynamics on the singular surface. We submit the reader to the Appendix B for further details.

[4] See the references [71, 87] for a classical treatment of cosmological solutions near the initial singularity in FRW cosmologies.

[5] We submit the reader to the Appendix B for a discussion of how to dealing with the singular points at the hypersurface $\Omega_\lambda + \Sigma^2 = 1$.

TABLE II: Critical points of the system (21)-(26) associated to expanding (contracting) solutions and their existence conditions. We use the notation s^* for an s-value such that $f(s^*)=0$ and s_c for an arbitrary s-value.

Label	Q	x	Ω_m	Ω_λ	Σ	s	Existence
$Q_1^\pm$	± 1	0	1	0	0	$s_c\in\mathbb{R}$	always
$Q_2^\pm(s^*)$	± 1	1	0	0	0	s^*	always
$Q_3^\pm(s^*)$	± 1	-1	0	0	0	s^*	always
$Q_4^\pm(s^*)$	± 1	$\pm\sqrt{\frac{3}{2}}\frac{\gamma}{s^*}$	$1-\frac{3\gamma}{s^{*2}}$	0	0	s^*	$s^{*2}\geq 3\gamma$
$Q_5^\pm(s^*)$	± 1	$\pm\frac{s^*}{\sqrt{6}}$	0	0	0	s^*	$s^{*2}\leq 6$
$Q_6^\pm$	± 1	0	0	0	-1	$s_c\in\mathbb{R}$	always
$Q_7^\pm$	± 1	0	0	0	1	$s_c\in\mathbb{R}$	always
$Q_8^\pm(s^*)$	± 1	$\cos u$	0	0	$\sin u$	s^*	always
$Q_9^\pm$	± 1	0	0	0	0	0	always
$Q_{10}^\pm$	± 1	0	0	$\Omega_{\lambda c}\in(0,1)$	0	0	always
$Q_{11}^\pm$	± 1	0	0	1	0	$s_c\in\mathbb{R}$	always
$Q_{12}^\pm(s^*)$	$\pm\frac{s^*}{2}$	$\pm\sqrt{\frac{2}{3}}$	0	0	0	s^*	$s^{*2}\leq 4$

The eigenvalues of the linear perturbation matrix evaluated at each of these critical points are displayed in the table III.

In section IV A are discussed the stability conditions for the expanding (contracting) solutions.

Additionally the system (22)-(26) admits eighteen classes of (curves of) fixed points corresponding to static solutions ($H=Q=0$). They are displayed in table IV. The eigenvalues of the linear perturbation matrix evaluated at each of these critical points are displayed in the table V. In section IV B are discussed the stability conditions and physical interpretation for this static solutions.

Summarizing the system (22)-(26) admits forty two classes of fixed points. For that reason this scenario has a rich cosmological behavior, including the transition from contracting to expanding solutions and vice versa. Also is possible the existence of bouncing solutions and a turnaround, and even cyclic behavior.

A. Stability of expanding (contracting) solutions

Now let us comment on the stability of the first order perturbations of (21)-(26) near the critical points showed in table II.

The line of singular points $Q_1^\pm$, although it is non-hyperbolic (actually, normally hyperbolic), they behave like a saddle point in the phase space of the RS model, since they have both nonempty stable and unstable manifolds (see the table III). The class Q_1^+ is the analogous to the line denoted by P_1 in [24].

The singular points $Q_2^\pm(s^*)$ and $Q_3^\pm(s^*)$ are non-hyperbolic (these points are related to $P_3^\pm$ investigated in [24]), however they behave as saddle points since they have both nonempty stable and unstable manifolds (see the table III).

The singular point $Q_4^+(s^*)$ is the analogous to P_4 in [24]. It is a stable node in the cases $0<\gamma\leq\frac{2}{9}, s^*<-\sqrt{3\gamma}, f'(s^*)<0$ or $\frac{2}{9}<\gamma<\frac{4}{3}, -\frac{2\sqrt{6}\gamma}{\sqrt{9\gamma-2}}\leq s^*<-\sqrt{3\gamma}, f'(s^*)<0$, or $0<\gamma\leq\frac{2}{9}, s^*>\sqrt{3\gamma}, f'(s^*)>0$, or $\frac{2}{9}<\gamma<\frac{4}{3}, \sqrt{3\gamma}<s^*\leq\frac{2\sqrt{6}\gamma}{\sqrt{9\gamma-2}}, f'(s^*)>0$.

It is a spiral stable point for either $\frac{2}{9}<\gamma<\frac{4}{3}, s^*<-\frac{2\sqrt{6}\gamma}{\sqrt{9\gamma-2}}, f'(s^*)<0$ or $\frac{2}{9}<\gamma<\frac{4}{3}, s^*>\frac{2\sqrt{6}\gamma}{\sqrt{9\gamma-2}}, f'(s^*)>0$.

In summary, it is stable for either $0<\gamma<\frac{4}{3}, s^*<-\sqrt{3\gamma}, f'(s^*)<0$ or $0<\gamma<\frac{4}{3}, s^*>\sqrt{3\gamma}, f'(s^*)>0$. Otherwise, it is a saddle point.

The singular point $Q_4^-(s^*)$ is a local source under the same conditions for which $Q_4^+(s^*)$ is stable.

TABLE III: Eigenvalues for the critical points in Table II. We use the notations $K^{\pm}(\gamma,s^*) = -\frac{3}{4}\left(2-\gamma\pm\frac{\sqrt{2-\gamma}\sqrt{s^{*2}(2-9\gamma)+24\gamma^2}}{s^*}\right)$, $L^{\pm}(\Omega_\lambda)=-\frac{1}{2}\left(3\pm\sqrt{12f(0)(\Omega_\lambda-1)+9}\right)$ and $M^{\pm}(s^*)=\frac{1}{4}\left(-s^*\pm\sqrt{64-15\,(s^*)^2}\right)$, where s^* denotes an s-value such that $f(s^*)=0$.

Label	Eigenvalues
$Q_1^{\pm}$	$\left\{(3\gamma-4)\epsilon,\frac{3}{2}(\gamma-2)\epsilon,\frac{3}{2}(\gamma-2)\epsilon,-3\gamma\epsilon,3\gamma\epsilon,0\right\}$
$Q_2^{\pm}(s^*)$	$\left\{2\epsilon,0,-6\epsilon,-3(\gamma-2)\epsilon,6\epsilon-\sqrt{6}s^*,-\sqrt{6}f'\left(s^*\right)\right\}$
$Q_3^{\pm}(s^*)$	$\left\{2\epsilon,0,-6\epsilon,-3(\gamma-2)\epsilon,6\epsilon\sqrt{6}s^*,\sqrt{6}f'\left(s^*\right)\right\}$
$Q_4^{\pm}(s^*)$	$\left\{-3\gamma\epsilon,\frac{3}{2}(\gamma-2)\epsilon,(3\gamma-4)\epsilon,\epsilon K^{+}(\gamma,s^*),\epsilon K^{-}(\gamma,s^*),-\frac{3\gamma\epsilon f'(s^*)}{s^*}\right\}$
$Q_5^{\pm}(s^*)$	$\left\{-\epsilon\left(s^*\right)^2,\frac{1}{2}\left(s^{*2}-6\right)\epsilon,\frac{1}{2}\left(s^{*2}-6\right)\epsilon,\left(s^{*2}-3\gamma\right)\epsilon,(s^{*2}-4)\epsilon,-\epsilon s^* f'\left(s^*\right)\right\}$
$Q_6^{\pm}$	$\{6\epsilon,6\epsilon,2\epsilon,0,0,3(2-\gamma)\epsilon\}$
$Q_7^{\pm}$	$\{6\epsilon,6\epsilon,2\epsilon,0,0,3(2-\gamma)\epsilon\}$
$Q_8^{\pm}(s^*)$	$\left\{0,6\epsilon-\sqrt{6}s\cos(u),-3(\gamma-2)\epsilon,2\epsilon,-6\epsilon,-\sqrt{6}\cos(u)f'(s)\right\}$
$Q_9^{\pm}$	$\left\{-4\epsilon,-3\epsilon,0,0,\frac{1}{2}\left(-\sqrt{9-12f(0)}3\right)\epsilon,\frac{1}{2}\left(\sqrt{9-12f(0)}-3\right)\epsilon\right\}$
$Q_{10}^{\pm}$	$\left\{-4\epsilon,-3\epsilon,0,-3\gamma\epsilon\Omega_\lambda,\epsilon L^{+}(\Omega_\lambda),\epsilon L^{-}(\Omega_\lambda)\right\}$
$Q_{11}^{\pm}$	$\{2(3\gamma-2)\epsilon,3(\gamma-1)\epsilon,3\gamma\epsilon,3\gamma\epsilon,3(\gamma-1)\epsilon,0\}$
$Q_{12}^{\pm}(s^*)$	$\left\{-2s^*\epsilon,-\frac{s^*\epsilon}{2},\frac{1}{2}(4-3\gamma)s^*\epsilon,\epsilon M^{+}(s^*),\epsilon M^{-}(s^*),-2\epsilon f'\left(s^*\right)\right\}$

The singular point $Q_5^+(s^*)$ is the analogous to P_5 in [24]. It is not hyperbolic for $s^*\in\left\{0,\pm\sqrt{6},\pm\sqrt{3\gamma},2\right\}$ or $f'(s^*)=0$.

In the hyperbolic case, $Q_5^+(s^*)$ is a stable node for either $0<\gamma\leq\frac{4}{3},-\sqrt{3\gamma}<s^*<0,f'\left(s^*\right)<0$ or $\frac{4}{3}<\gamma\leq 2,-2<s^*<0,f'\left(s^*\right)<0$ or $0<\gamma\leq\frac{4}{3},0<s^*<\sqrt{3\gamma},f'\left(s^*\right)>0$ or $\frac{4}{3}<\gamma\leq 2,0<s^*<2,f'\left(s^*\right)>0$; otherwise, it is a saddle point.

The singular point $Q_5^-(s^*)$ is a local source under the same conditions for which $Q_5^+(s^*)$ is stable.

For evaluating the Jacobian matrix, and obtaining their eigenvalues at the singular points $Q_6^{\pm}$, $Q_7^{\pm}$ and $Q_{11}^{\pm}$ we need to take the appropriate limits. The details on the stability analysis for Q_6^+ and Q_{11}^+ are offered in the Appendix (B). The analysis of Q_7^+ is essentially the same as for Q_6^+.

The circles of critical points $Q_8^{\pm}$ are non-hyperbolic. Both solutions represents transient states in the evolution of the universe.

The singular point $Q_9^{\pm}$ and the line of critical points $Q_{10}^{\pm}$ are non-hyperbolic (Q_9^+ is the analogous of P_{10} for $\Omega_\lambda=0$ while Q_{10}^+ is the analogous of P_{10} for $\Omega_\lambda\neq 0$ in [24]). In the Appendix A we analyze the stability of Q_9^+ by introducing a local set of coordinates adapted to the singular points. The non-hyperbolic fixed point Q_9^-, behaves as a past attractor.

The singular point Q_{11}^+ is the analogous of P_{11} in [24], this point represents a 1D set of singular points such that $(\Omega_\lambda=1)$, parameterized by the values of s_c. It is normally hyperbolic.

The singular points $Q_{12}^{\pm}(s^*)$ are non-hyperbolic for $s^*=0,\pm 2$, $\gamma=4/3$ or $f'(s^*)=0$. $Q_{12}^+(s^*)$ is a stable node $\frac{4}{3}<\gamma\leq 2,2<s^*\leq\frac{8}{\sqrt{15}},f'\left(s^*\right)>0$, a stable spiral for $\frac{4}{3}<\gamma\leq 2,s^*>\frac{8}{\sqrt{15}},f'\left(s^*\right)>0$. In summary, $Q_{12}^+(s^*)$ is stable for $\frac{4}{3}<\gamma\leq 2,s^*>2,f'\left(s^*\right)>0$. Otherwise it is a saddle point. $Q_{12}^-(s^*)$ is unstable for the same conditions for which $Q_{12}^+(s^*)$ is stable.

Summarizing, the possible late time attractors are:

- $Q_4^+(s^*)$ is stable for either $0<\gamma<\frac{4}{3},s^*<-\sqrt{3\gamma},f'\left(s^*\right)<0$ or $0<\gamma<\frac{4}{3},s^*>\sqrt{3\gamma},f'\left(s^*\right)>0$;
- $Q_5^+(s^*)$ is stable for either $0<\gamma\leq\frac{4}{3},-\sqrt{3\gamma}<s^*<0,f'\left(s^*\right)<0$ or $\frac{4}{3}<\gamma\leq 2,-2<s^*<0,f'\left(s^*\right)<0$ or $0<\gamma\leq\frac{4}{3},0<s^*<\sqrt{3\gamma},f'\left(s^*\right)>0$ or $\frac{4}{3}<\gamma\leq 2,0<s^*<2,f'\left(s^*\right)>0$;
- Q_9^+ and the line Q_{10}^+ are stable for $f(0)\geq 0$. However, if we include the variable $y=\frac{V}{3H^2}$, in the analysis, the solutions associated to the line of fixed points Q_{10}^+ are unstable to perturbations along the y-direction.

TABLE IV: Critical points of the system (21)-(26) representing static solutions and their existence conditions. We use the notation s^* for an s-value such that $f(s^*)=0$ and s_c for an arbitrary s-value. We use the notation $f_1(\gamma,\Omega_\lambda,\Sigma)=\frac{2\left(3\Sigma^2-2\right)\left(\Omega_\lambda+\Sigma^2-1\right)}{3\gamma\left(\Omega_\lambda-\Sigma^2+1\right)}$, $f_2(\Omega_\lambda,\Sigma)=\frac{4-6\Sigma^2}{-3\Sigma^2+3\Omega_\lambda+3}$, $f_3(\gamma,\Sigma)=2\left(1-\Sigma^2\right)-\frac{\frac{4}{3}-2\Sigma^2}{\gamma}$, $f_4(\gamma,\Sigma)=\frac{\frac{4}{3}-2\Sigma^2}{\gamma}+\Sigma^2-1$, $f_5(\Sigma)=\frac{3\Sigma^2-2}{3\Sigma^2-3}$, $f_6(\gamma,x,\Sigma)=\frac{-6\Sigma^2+6x^2-\sqrt{4\left(3(\gamma-1)\Sigma^2-3\gamma+2\right)^2+9(\gamma-2)^2x^4+12(\gamma-2)x^2\left(3(\gamma+1)\Sigma^2-3\gamma-2\right)}+4}{6\gamma}$, $f_7(\gamma,x)=\frac{6x^2+\sqrt{4(2-3\gamma)^2+9(\gamma-2)^2x^4+12((4-3\gamma)\gamma+4)x^2}+4}{6\gamma}$

Label	Q	x	Ω_m	Ω_λ	Σ	s	Existence
$E_1^\pm$	0	0	0	$\Omega_{\lambda c}\in\left[0,\frac{1}{3}\right]$	$\pm\sqrt{\frac{2}{3}}$	0	$0<\gamma\leq 2$
E_2	0	0	0	$\cos^2 u$	$\sin u$	0	$\Omega_{\lambda c}\notin\left\{\pm1,\pm\sqrt{\frac{2}{3}}\right\},0<\gamma\leq 2,0\leq u\leq 2\pi$
E_3	0	0	$f_1(\gamma,\Omega_{\lambda_c},\Sigma_c)$	$\Omega_{\lambda c}\in\left[0,1-\Sigma_c^2\right)$	$\Sigma_c\in\left(-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)$	0	$f_2(\Omega_{\lambda_c},\Sigma_c)\leq\gamma\leq 2$
E_4	0	0	$f_3(\gamma,\Sigma_c)$	$f_4(\gamma,\Sigma_c)$	Σ_c	s^*	$-1\leq\Sigma_c\leq-\sqrt{\frac{2}{3}},0<\gamma\leq 2$ or
							$-\sqrt{\frac{2}{3}}<\Sigma_c<\sqrt{\frac{2}{3}},f_5(\Sigma_c)\leq\gamma\leq 2$ or
							$\sqrt{\frac{2}{3}}\leq\Sigma_c\leq 1,0<\gamma\leq 2.$
E_5	0	x_c	$1-\Sigma_c^2-x_c^2-\Omega_{\lambda c}$	$-\frac{x_c^2}{2}+f_6(\gamma,x_c,\Sigma_c)$	Σ_c	s^*	$x_c^2+\Sigma_c^2<\frac{2}{3},0<\gamma\leq 2-\frac{2}{3(1-\Sigma_c^2-x_c^2)}$
$E_6^\pm$	0	$\pm\sqrt{\frac{2}{3}-\Sigma_c^2}$	0	0	Σ_c	0	$-\sqrt{\frac{2}{3}}<\Sigma_c<\sqrt{\frac{2}{3}},0\leq\gamma\leq 2$
$E_7^\pm$	0	$\pm\frac{\Sigma_c}{\sqrt{3}}$	0	$1-2\Sigma_c^2$	Σ_c	0	$\Sigma_c^2<\frac{1}{2},\Sigma_c\neq 0,0\leq\gamma\leq 2$
E_8	0	x_c	$1-x_c^2-\Omega_{\lambda c}$	$-\frac{x_c^2}{2}+f_7(\gamma,x_c)$	0	s^*	$x_c^2<\frac{2}{3},0<\gamma\leq\frac{4-6x_c^2}{3-3x_c^2}$
E_9	0	0	$\frac{4(1-\Omega_{\lambda c})}{3\gamma(\Omega_{\lambda c}+1)}$	$\Omega_{\lambda c}$	0	0	$0<\gamma\leq\frac{2}{3},\Omega_{\lambda c}=1$ or
							$\frac{2}{3}<\gamma\leq\frac{4}{3},\frac{4}{3\gamma}-1\leq\Omega_{\lambda c}\leq 1$, or
							$\frac{4}{3}<\gamma\leq 2,0\leq\Omega_{\lambda c}\leq 1.$
$E_{10}^\pm$	0	$\pm\sqrt{\frac{2-3\gamma}{3(2-\gamma)}}\Sigma_c$	$\frac{4\Sigma_c^2}{3(2-\gamma)}$	$1-2\Sigma_c^2$	Σ_c	s^*	$\gamma\neq\frac{2}{3},\Sigma_c=0$ or
							$0\leq\gamma<\frac{2}{3},0<\Sigma_c^2\leq\frac{1}{2}.$
E_{11}	0	0	$\frac{2\Sigma_c^2}{3\gamma}$	$1-2\Sigma_c^2$	Σ_c	0	$0<\gamma<\frac{2}{3},\Sigma_c=0$ or
							$\frac{2}{3}\leq\gamma\leq 2,\Sigma_c^2\leq\frac{1}{2}.$
E_{12}	0	0	$\frac{2(2-3\Sigma_c^2)}{3\gamma}$	0	Σ_c	0	$0<\gamma<\frac{4}{3},1-\frac{2}{3(2-\gamma)}\leq\Sigma_c^2\leq\frac{2}{3}$ or $\frac{4}{3}\leq\gamma\leq 2,\Sigma_c^2\leq\frac{2}{3}.$
$E_{13}^\pm$	0	$\pm\sqrt{\frac{4-3\gamma}{3(2-\gamma)}}$	$\frac{2}{3(2-\gamma)}$	0	0	s^*	$0<\gamma\leq\frac{4}{3}.$

TABLE V: Eigenvalues for the critical points in Table IV. We use the notations $g_1(\gamma,\Omega_\lambda,\Sigma) = \frac{\sqrt{-3\gamma\left(3\Sigma^2-2\right)\left(-\Sigma^2+\Omega_\lambda+1\right)^2+18\Sigma^6-44\Sigma^4+2\Sigma^2(\Omega_\lambda(9\Omega_\lambda+2)+17)-8\left(\Omega_\lambda^2+1\right)}}{-\Sigma^2+\Omega_\lambda+1}$, $g_2(\gamma,\Omega_\lambda,\Sigma) = \frac{\sqrt{f(0)}\sqrt{\Sigma^2+\Omega_\lambda-1}\sqrt{3\gamma\left(-\Sigma^2+\Omega_\lambda+1\right)+6\Sigma^2-4}}{\sqrt{\gamma}\sqrt{-\Sigma^2+\Omega_\lambda+1}}$,
$g_3(\gamma,x,\Omega_\lambda,\Sigma) = -\frac{9}{2}\left(\gamma^2(\Omega_\lambda-1)\Omega_\lambda+(\gamma-2)^2\Sigma^4-(\gamma-2)\Sigma^2(2\gamma\Omega_\lambda+\gamma-2)\right) - \frac{9(\gamma-2)^2x^4\left(\Sigma^4-2\Sigma^2(\Omega_\lambda+1)+\Omega_\lambda^2+1\right)}{2\left(\Sigma^2+\Omega_\lambda-1\right)^2} - \frac{9(\gamma-2)x^2\left(-\Omega_\lambda\left(4(\gamma-1)\Sigma^2+\gamma+2\right)+(\gamma-2)\left(2\Sigma^4-3\Sigma^2+1\right)+2\gamma\Omega_\lambda^2\right)}{2\left(\Sigma^2+\Omega_\lambda-1\right)}$,
$g_4(\gamma,x,\Omega_\lambda) = \frac{3\sqrt{-\gamma^2(\Omega_\lambda-1)^3\Omega_\lambda-(\gamma-2)^2x^4\left(\Omega_\lambda^2+1\right)-(\gamma-2)x^2(\Omega_\lambda-1)\left(2\gamma\Omega_\lambda^2-(\gamma+2)\Omega_\lambda+\gamma-2\right)}}{\sqrt{2}(\Omega_\lambda-1)}$
$g_5(\gamma,\Omega_\lambda) = \frac{\sqrt{6\gamma(\Omega_\lambda+1)^2-8\left(\Omega_\lambda^2+1\right)}}{\Omega_\lambda+1}$, $g_6(\gamma,\Omega_\lambda) = \frac{\sqrt{f(0)}\sqrt{\Omega_\lambda-1}\sqrt{3\gamma(\Omega_\lambda+1)-4}}{\sqrt{\gamma}\sqrt{\Omega_\lambda+1}}$,
$g_7(\gamma,\Sigma) = \sqrt{2}\sqrt{\gamma\left(9\Sigma^2-6\right)-4\Sigma^2+4}$, $g_8(\gamma,\Sigma) = \sqrt{\gamma\left(6-9\Sigma^2\right)+18\Sigma^2-8}$,
and $g_9(\gamma,\Sigma) = \frac{\sqrt{f(0)}\sqrt{3(\gamma-2)\Sigma^2-3\gamma+4}}{\sqrt{\gamma}}$.
$\lambda_1(\epsilon,\Sigma_c), \lambda_2(\epsilon,\Sigma_c), \lambda_3(\epsilon,\Sigma_c)$ and $\lambda_4(\epsilon,\Sigma_c)$ are the roots of the equation
$\lambda^4+\lambda^3\sqrt{4-6\Sigma_c^2}\epsilon f'(0)+\lambda^2\left[f(0)-4\right]-4\lambda\sqrt{4-6\Sigma_c^2}\epsilon f'(0)-6f(0)\Sigma_c^2=0$.
$\mu_1(\epsilon,\Sigma_c), \mu_2(\epsilon,\Sigma_c), \mu_3(\epsilon,\Sigma_c)$ and $\mu_4(\epsilon,\Sigma_c)$ are the roots of the equation $\mu^4+\sqrt{2}\mu^3\Sigma_c\epsilon f'(0)+\mu^2\left[2(f(0)+4){\Sigma_c}^2-8\right]+8\sqrt{2}\mu\Sigma_c\left(\Sigma_c^2-1\right)\epsilon f'(0)+4f(0)\Sigma_c^2\left(\Sigma_c^2-2\right)=0$.

Labels	Eigenvalues
$E_1^\pm$	$\left\{-2,2,0,0,-\sqrt{f(0)}\sqrt{3\Omega_{\lambda c}-1},\sqrt{f(0)}\sqrt{3\Omega_{\lambda c}-1}\right\}$
E_2	$\left\{0,0,0,0,-\frac{3}{2}\sqrt{1-\cos(4u)},\frac{3}{2}\sqrt{1-\cos(4u)}\right\}$
E_3	$\{0,0,-g_1(\gamma,\Omega_{\lambda c},\Sigma_c),g_1(\gamma,\Omega_{\lambda c},\Sigma_c),-g_2(\gamma,\Omega_{\lambda c},\Sigma_c),g_2(\gamma,\Omega_{\lambda c},\Sigma_c)\}$
E_4	$\left\{0,0,0,0,-\sqrt{-9\gamma^2+9(\gamma-2)(\gamma-1)\Sigma_c^2+18\gamma-8},\sqrt{-9\gamma^2+9(\gamma-2)(\gamma-1)\Sigma_c^2+18\gamma-8}\right\}$
E_5	$\left\{0,0,-\sqrt{g_3(\gamma,x_c,\Omega_{\lambda c},\Sigma_c)},\sqrt{g_3(\gamma,x_c,\Omega_{\lambda c},\Sigma_c)},-\sqrt{6}s^*x_c,-\sqrt{6}x_cf'\left(s^*\right)\right\}$
E_6^ϵ	$\{0,0,\lambda_1(\epsilon,\Sigma_c),\lambda_2(\epsilon,\Sigma_c),\lambda_3(\epsilon,\Sigma_c),\lambda_4(\epsilon,\Sigma_c)\}$
E_7^ϵ	$\{0,0,\mu_1(\epsilon,\Sigma_c),\mu_2(\epsilon,\Sigma_c),\mu_3(\epsilon,\Sigma_c),\mu_4(\epsilon,\Sigma_c)\}$
E_8	$\left\{0,0,-g_4(\gamma,x_c,\Omega_{\lambda c}),g_4(\gamma,x_c,\Omega_{\lambda c}),-\sqrt{6}s^*x_c,-\sqrt{6}x_cf'\left(s^*\right)\right\}$
E_9	$\{0,0,-g_5(\gamma,\Omega_{\lambda c}),g_5(\gamma,\Omega_{\lambda c}),-g_6(\gamma,\Omega_{\lambda c}),g_6(\gamma,\Omega_{\lambda c})\}$
E_{10}^ϵ	$\left\{0,0,-\epsilon\sqrt{2}\sqrt{\frac{3\gamma-2}{\gamma-2}}s^*\Sigma_c,g_7(\gamma,\Sigma_c),-g_7(\gamma,\Sigma_c),-\epsilon\sqrt{2}\sqrt{\frac{3\gamma-2}{\gamma-2}}\Sigma_cf'\left(s^*\right)\right\}$
E_{11}	$\left\{0,0,-\sqrt{\gamma\left(6-9\Sigma_c^2\right)+10\Sigma_c^2-4},\sqrt{\gamma\left(6-9\Sigma_c^2\right)+10\Sigma_c^2-4},-\frac{\sqrt{2-3\gamma}\sqrt{f(0)}\Sigma_c}{\sqrt{\gamma}},\frac{\sqrt{2-3\gamma}\sqrt{f(0)}\Sigma_c}{\sqrt{\gamma}}\right\}$
E_{12}	$\{0,0,-g_8(\gamma,\Sigma_c),g_8(\gamma,\Sigma_c),-g_9(\gamma,\Sigma_c),g_9(\gamma,\Sigma_c)\}$
E_{13}^ϵ	$\left\{0,0,\sqrt{4-3\gamma},-\sqrt{4-3\gamma},-\frac{\sqrt{2}\sqrt{4-3\gamma}s^*\epsilon}{\sqrt{2-\gamma}},-\sqrt{2}\sqrt{\frac{3\gamma-4}{\gamma-2}}\epsilon f'\left(s^*\right)\right\}$

- Q_{11}^- is stable for $\gamma>1$.
- $Q_{12}^+(s^*)$ is stable for $\frac{4}{3}<\gamma\leq 2, s^*>2, f'\left(s^*\right)>0$;

whereas, the possible past attractors are:

- $Q_4^-(s^*)$ is a source for either $0<\gamma<\frac{4}{3}, s^*<-\sqrt{3\gamma}, f'\left(s^*\right)<0$ or $0<\gamma<\frac{4}{3}, s^*>\sqrt{3\gamma}, f'\left(s^*\right)>0$;
- $Q_5^-(s^*)$ is a source for either $0<\gamma\leq\frac{4}{3}, -\sqrt{3\gamma}<s^*<0, f'\left(s^*\right)<0$ or $\frac{4}{3}<\gamma\leq 2, -2<s^*<0, f'\left(s^*\right)<0$ or $0<\gamma\leq\frac{4}{3}, 0<s^*<\sqrt{3\gamma}, f'\left(s^*\right)>0$ or $\frac{4}{3}<\gamma\leq 2, 0<s^*<2, f'\left(s^*\right)>0$;
- Q_9^- and the line Q_{10}^- are unstable for $f(0)\geq 0$.
- Q_{11}^+ is unstable for $\gamma>1$.
- $Q_{12}^-(s^*)$ is source for $\frac{4}{3}<\gamma\leq 2, s^*>2, f'\left(s^*\right)>0$;

B. Stability of static solutions

Having commented the stability issue for expanding (contracting) solutions, let us examine the stability of static solutions. As we commented before, the system (22)-(26) admits eighteen classes of (curves of) fixed points corresponding to static solutions, i.e., having $Q=0$, that is with $H=0$. Their coordinates in the phase space and their existence conditions are displayed in table IV.

Now let us comment on the stability of the first order perturbations of (21)-(26) near the critical points showed in table IV.

The line of fixed points E_1 is non-hyperbolic. However, the points located at the curve behaves as saddle points since its matrix of perturbations admits at least two real eigenvalues of different signs.

The one-parametric line of fixed points E_2 has two real eigenvalues of different signs provided $\cos(4u)\neq 1$. In this case, although non-hyperbolic, it behaves as a set of saddle points. For $u\in\left\{\frac{\pi}{2},\frac{3\pi}{2}\right\}$ all the eigenvalues are zero. In this case we need to resort to a numerical elaboration.

$g_1(\gamma,\Omega_\lambda,\Sigma)$ is always real valued for the allowed values of the phase space variables and the allowed range for the free parameters. Then, although non-hyperbolic, the 2D set of fixed points E_3 is of saddle type.

The eigenvalues associated to the one-parametric curve of fixed points E_4 are always reals for the allowed values of the phase space variables and the allowed range for the free parameters. Thus, although non-hyperbolic, it behaves as a saddle point.

For the allowed values of the phase space variables and the allowed range for the free parameters $g_3(\gamma,x_c,\Omega_{\lambda c},\Sigma_c)\geq 0$. If it is strictly positive, then the fixed points located in the 2D invariant set E_5 behaves as saddle points.

The eigenvalues of the perturbation matrix associated to E_6^ϵ are $\{0,0,\lambda_1(\epsilon,\Sigma_c),\lambda_2(\epsilon,\Sigma_c),\lambda_3(\epsilon,\Sigma_c),\lambda_4(\epsilon,\Sigma_c)\}$ where $\lambda_1(\epsilon,\Sigma_c),\lambda_2(\epsilon,\Sigma_c),\lambda_3(\epsilon,\Sigma_c)$ and $\lambda_4(\epsilon,\Sigma_c)$ are the roots of the polynomial equation with real coefficients:

$$\lambda^4+\lambda^3\sqrt{4-6\Sigma_c^2}\epsilon f'(0)+\lambda^2\left[f(0)-4\right]-4\lambda\sqrt{4-6\Sigma_c^2}\epsilon f'(0)-6f(0)\Sigma_c^2=0. \tag{31}$$

- For $f'(0)\neq 0, f(0)<0$, equation (31) has only two changes of signs in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there is zero or two positive roots of (31). Substituting λ by $-\lambda$ in (31) and applying the same rule we have zero or two negative roots of (31) for $f'(0)\neq 0, f(0)<0$. If all of them are complex conjugated, we need to resort to numerical investigation. If none of them are complex conjugated, then the curve of fixed points E_6^ϵ consists of saddle points.

- For $\epsilon f'(0)>0, f(0)>0$, equation (31) has only one change of sign in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there is one positive root of (31). Substituting λ by $-\lambda$ in (31) and applying the same rule we have three negative roots of (31) for $\epsilon f'(0)>0, f(0)>0$. In summary, for $\epsilon f'(0)>0, f(0)>0$, at least two eigenvalues of the linear perturbation matrix of E_6^+ are real of different signs. In this case the curve of fixed points E_6^ϵ consists of saddle points.

- For $\epsilon f'(0)<0, f(0)>0$, equation (31) has three changes of sign in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there are three positive roots of (31). Substituting λ by $-\lambda$ in (31) and applying the same rule we have only one negative root of (31) for $\epsilon f'(0)<0, f(0)>0$. In this case the curve of fixed points E_6^ϵ consists of saddle points.

- For $f'(0)=f(0)=0$ the non null eigenvalues are $-2,2$; for $f'(0)=0, f(0)\neq 0$, the non null eigenvalues are $\pm\frac{\sqrt{-\sqrt{24f(0)\Sigma_c^2+(f(0)-4)^2}-f(0)+4}}{\sqrt{2}},\pm\frac{\sqrt{\sqrt{24f(0)\Sigma_c^2+(f(0)-4)^2}-f(0)+4}}{\sqrt{2}}$; and for $f'(0)\neq 0, f(0)=0$, the non null eigenvalues are $-2,2,-\epsilon\sqrt{4-6\Sigma_c^2}f'(0)$. Thus, in both cases $E_6^\pm$ consists of saddle points.

The eigenvalues of the perturbation matrix associated to E_7^ϵ are $\{0,0,\mu_1(\epsilon,\Sigma_c),\mu_2(\epsilon,\Sigma_c),\mu_3(\epsilon,\Sigma_c),\mu_4(\epsilon,\Sigma_c)\}$ where $\mu_1(\epsilon,\Sigma_c),\mu_2(\epsilon,\Sigma_c),\mu_3(\epsilon,\Sigma_c)$ and $\mu_4(\epsilon,\Sigma_c)$ are the roots of the polynomial equation with real coefficients:

$$\mu^4+\sqrt{2}\mu^3\Sigma_c\epsilon f'(0)+\mu^2\left[2(f(0)+4){\Sigma_c}^2-8\right]+8\sqrt{2}\mu\Sigma_c\left(\Sigma_c^2-1\right)\epsilon f'(0)+4f(0)\Sigma_c^2\left(\Sigma_c^2-2\right)=0. \tag{32}$$

- For $\epsilon f'(0)<0, f(0)<0, -\frac{1}{\sqrt{2}}<\Sigma_c<0$ or $\epsilon f'(0)>0, f(0)<0, 0<\Sigma_c<\frac{1}{\sqrt{2}}$, or $\epsilon f'(0)<0, f(0)<0, 0<\Sigma_c<\frac{1}{\sqrt{2}}$, or $\epsilon f'(0)>0, f(0)<0, -\frac{1}{\sqrt{2}}<\Sigma_c<0$, the equation (32) has only two changes of signs in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there is zero or two positive roots of (32). Substituting μ by $-\mu$ in (32) and applying the same rule we have zero or two negative roots of (32) for the same values of the free parameters. If all of them are complex conjugated, we need to resort to numerical investigation. If none of them are complex conjugated, then the curve of fixed points E_7^ϵ consists of saddle points.

- For $f(0)>0, \epsilon f'(0)>0, 0<\Sigma_c<\frac{1}{\sqrt{2}}$ or $f(0)>0, \epsilon f'(0)<0, \frac{1}{\sqrt{2}}<\Sigma_c<0$, equation (32) has only one change of sign in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there is one positive root of (32). Substituting μ by $-\mu$ in (32) and applying the same rule we have three negative roots of (32) for the same values of the parameters. In this case the curve of fixed points E_7^ϵ consists of saddle points.
- For $f(0)>0, 0<\Sigma_c<\frac{1}{\sqrt{2}}, \epsilon f'(0)<0$ or $f(0)>0, -\frac{1}{\sqrt{2}}<\Sigma_c<0, \epsilon f'(0)>0$ equation (32) has three changes of sign in the sequence of its coefficients. Hence, using the Descartes's rule, we conclude that for this range of the parameters, there are three positive roots of (32). Substituting μ by $-\mu$ in (32) and applying the same rule we have only one negative root of (32) for the same values of the parameters. In this case the curve of fixed points E_7^ϵ consists of saddle points.
- For $f'(0)=f(0)=0$ the non null eigenvalues are $2\sqrt{2}\sqrt{1-\Sigma_c^2}, -2\sqrt{2}\sqrt{1-\Sigma_c^2}$; for $f'(0)=0, f(0)\neq 0$, the non null eigenvalues are $\pm\sqrt{-(f(0)+4)\Sigma_c^2-\sqrt{(f(0)(f(0)+4)+16)\Sigma_c^4-32\Sigma_c^2+16}+4}$, and $\pm\sqrt{-(f(0)+4)\Sigma_c^2+\sqrt{(f(0)(f(0)+4)+16)\Sigma_c^4-32\Sigma_c^2+16}+4}$; and for $f'(0)\neq 0, f(0)=0$, the non null eigenvalues are $-\sqrt{2}\Sigma_c\epsilon f'(0), 2\sqrt{2-2\Sigma_c^2}, -2\sqrt{2-2\Sigma_c^2}$. Thus, in both cases $E_7^\pm$ consists of saddle points.

Observe that $g_4(\gamma, x, \Omega_\lambda)$ is always real valued for the allowed values of the phase space variables and the allowed range for the free parameters. Then, although non-hyperbolic, E_8 behaves as a set of saddle points.

When $\Omega_{\lambda c}\neq 1$, $g_5(\gamma,\Omega_{\lambda c})$ is real valued, in this case the fixed points in the line E_9 behaves as saddle points.

For $\Sigma_c\neq 0$, $g_7(\gamma,\Omega_{\lambda c})$ is real valued, in this case the fixed points in the line E_{10} behaves as saddle points.

For $\Sigma_c\neq 0$, the linear perturbation matrix evaluated at E_{11} has at least two real eigenvalues of different signs, thus, the fixed points in the line E_{11} behaves as saddle points.

Observe that the line E_9 and the line E_{11} contains the special point with coordinates $F:(Q=0, x=0, \Omega_m=0, \Omega_\lambda=1, \Sigma=0)$ in the first case when $\Omega_{\lambda c}\to 1$ whereas in the second case for $\Sigma_c\to 0$. Taking in both cases the proper limits we have that the eigenvalues of the linearization for F are $\left\{0,0,-\sqrt{6\gamma-4},\sqrt{6\gamma-4},0,0\right\}$. Hence, for $\gamma>\frac{2}{3}$, F is of saddle type, whereas for $\gamma<\frac{2}{3}$, there are two purely imaginary eigenvalues and the rest of the eigenvalues are zero. In this case we cannot say anything about its stability from the linearization and we need to resort to numerical inspection.

Observe that $g_8(\gamma,\Sigma)$ is real valued for the allowed values of the phase space variables and the allowed range for the free parameters. Thus, all the points located at the line E_{12} behaves as saddle points.

For $0<\gamma<\frac{4}{3}$, $E_{13}^\pm$ behaves as a saddle point.

Summarizing, our system admits a large class of static solutions that are proved to be typically of saddle like. This kind of solutions are important as intermediated stages in the evolution of the universe allowing to the transition from expanding to contracting models.

V. PHYSICAL DISCUSSION

In order to characterize the cosmological solutions associated to the singular points it will be helpful to have the important observational parameters in terms of the state variables. The dimensionless scalar field energy density parameter $\Omega_\phi=\frac{\rho_\phi}{3H^2}$, the equation of state parameter $\omega_\phi=\frac{p_\phi}{\rho_\phi}$, and deceleration parameter $q=-\left(1+\frac{\dot{H}}{H^2}\right)$. They read:

$$\Omega_\phi=\frac{1-\Omega_\lambda-\Omega_m-\Sigma^2}{Q^2}, \qquad \omega_\phi=-1+\frac{2x^2}{1-\Omega_\lambda-\Omega_m-\Sigma^2} \tag{33}$$

$$\omega_{\mathrm{eff}}\equiv\frac{p_T}{\rho_T}=\frac{2x^2+(\gamma+1)\Omega_m+\Omega_\lambda+\Sigma^2-1}{1-\Omega_\lambda-\Sigma^2} \tag{34}$$

and

$$q=-1+\frac{3}{Q^2}\left(1+\frac{2\Omega_\lambda}{1-\Omega_\lambda+\Sigma^2}\right)\left(x^2+\frac{\gamma\Omega_m}{2}\right)+2\left(1-\frac{1}{Q^2}\right)+\frac{3\Sigma^2}{Q^2}$$

Observe that these parameters, but Ω_ϕ, blow up as the singular surfaces $\Omega_\lambda+\Omega_m+\Sigma^2=1$ and $\Omega_\lambda+\Sigma^2=1$ are approached. Thus, we need to take the appropriate limits for evaluating at the singular points $Q_6^\pm, Q_7^\pm$ and $Q_{11}^\pm$. In

TABLE VI: Values of the observable parameters Ω_ϕ, ω_ϕ, $\omega_{\rm eff}$, and q evaluated at the critical points of the system (21)-(26). We use the notations s^* for an s-value such that $f(s^*)=0$.

P_i	Ω_ϕ	ω_ϕ	$\omega_{\rm eff}$	q
$Q_1^\pm$	0	arbitrary	γ	$\frac{3\gamma-2}{2}$
$Q_2^\pm(s^*), Q_3^\pm(s^*)$	1	1	1	2
$Q_4^\pm(s^*)$	$\frac{3\gamma}{s^{*2}}$	$\gamma-1$	$\gamma-\frac{3\gamma}{s^{*2}}$	$\frac{3\gamma}{2}-1$
$Q_5^\pm(s^*)$	1	$\frac{1}{3}\left(s^{*2}-3\right)$	$\frac{1}{3}\left(s^{*2}-3\right)$	$\frac{1}{2}\left(s^{*2}-2\right)$
$Q_6^\pm$	0	arbitrary	arbitrary	2
$Q_7^\pm$	0	arbitrary	arbitrary	2
$Q_8^\pm(s^*)$	$\cos^2(u)$	1	1	2
$Q_9^\pm$	1	-1	-1	-1
$Q_{10}^\pm$	$1-\Omega_\lambda$	-1	-1	-1
$Q_{11}^\pm$	0	arbitrary	arbitrary	-1
$Q_{12}^\pm(s^*)$	$\frac{4}{s^{*2}}$	$\frac{1}{3}$	$\frac{1}{3}$	1

table VI we present the values of the observable parameters Ω_ϕ, ω_ϕ, $\omega_{\rm eff}$, and q, evaluated at the critical points of the system (21)-(26) that are associated to expanding (contracting) solutions. Static solutions were discussed in section IV B.

Observe that, since the variable Q defined in (17) is the Hubble scalar divided by a positive constant, $Q>0$ corresponds to an expanding universe, while $Q<0$ to a contracting one. Furthermore, as usual, for an expanding universe $q<0$ corresponds to accelerating expansion and $q>0$ to decelerating expansion, while for a contracting universe $q<0$ corresponds to decelerating contraction and $q>0$ to accelerating contraction. Lastly, critical points with $\Sigma=0$ correspond to isotropic universes.

Now, let us comment on the physical interpretation of the critical points Q_i that are associated to expanding (contracting) solutions.

The line of singular points $Q_1^\pm$ represent a matter-dominated solution ($\Omega_m=1$), as expected they are transient stages in the evolution of the universe.

The singular points $Q_2^\pm(s^*)$ and $Q_3^\pm(s^*)$ are solutions dominated by the kinetic energy of the scalar field. At these solutions, the scalar field mimics stiff matter, and $\rho_\phi\propto a^{-6}$ where a denotes the scale factor. This solutions represent transient states in the evolution of the universe.

The singular point $Q_4^+(s^*)$ represent matter-scalar field scaling solutions that are relevant attractors for either $0<\gamma<\frac{4}{3}, s^*<-\sqrt{3\gamma}, f'(s^*)<0$ or $0<\gamma<\frac{4}{3}, s^*>\sqrt{3\gamma}, f'(s^*)>0$. They satisfy $\Omega_\phi\sim\Omega_m$ thus they are important for solving or alleviating the coincidence problem. The corresponding solutions are accelerating only for $\gamma<\frac{2}{3}$. Thus, for standard matter ($\gamma>1$), that is satisfying the usual energy conditions, they cannot represent accurately the present universe, since they are decelerating solutions. However, the most interesting feature is that the singular point $Q_4^-(s^*)$, which corresponds to a contracting universe (since $\mathrm{sgn}(Q)=\mathrm{sgn}(H)=-1$), is a local source under the same conditions for which $Q_4^+(s^*)$, which corresponds to an expanding universe, is stable. Therefore, in the scenario at hand the transition from contracting to expanding universes is possible, since for the same values of the parameters we have a contracting local source and an expanding local attractor. Such a behavior has very important physical implications.

The singular point $Q_5^+(s^*)$ is the analogous to P_5 in [24]. It represents a scalar-field dominated solution ($\Omega_\phi=1$) that is a relevant attractor for either $0<\gamma\leq\frac{4}{3}, -\sqrt{3\gamma}<s^*<0, f'(s^*)<0$ or $\frac{4}{3}<\gamma\leq 2, -2<s^*<0, f'(s^*)<0$ or $0<\gamma\leq\frac{4}{3}, 0<s^*<\sqrt{3\gamma}, f'(s^*)>0$ or $\frac{4}{3}<\gamma\leq 2, 0<s^*<2, f'(s^*)>0$. These solutions can represent accurately the late-time universe if additionally $s^{*2}<2$, in which case they represent accelerating solutions. As before, the singular point $Q_5^-(s^*)$ is a local source under the same conditions for which $Q_5^+(s^*)$ is stable. This means that we have a large probability to have a transition from contracting to expanding solutions. This is one of the advantages of the present model.

The singular point $Q_9^\pm$ represents an standard 4D cosmological solution whereas $Q_{10}^\pm$ represent solutions with 5D-corrections. In both cases, from the relationship between y and Ω_λ follows that this solution is dominated by the

potential energy of the scalar field $\rho_T = V(\phi)$; that is, it is de Sitter-like solution ($\omega_\phi = -1$). In this case the Friedmann equation can be expressed as

$$3H^2 = V\left(1 + \frac{V}{2\lambda}\right) \tag{35}$$

In the early universe, where $\lambda \ll V$, the expansion rate of the universe for the RS model differs from the general relativity predictions

$$\frac{H_{RS}}{H_{GR}} = \sqrt{\frac{V}{2\lambda}} \tag{36}$$

Contracting de Sitter solutions are associated to the non-hyperbolic fixed point Q_9^-, and it corresponds to the early-time universe since it behaves as a local source in the phase-space.

The singular points Q_{11}^{ϵ} represent a 1D set of singular points, corresponding to isotropic solutions with 5D corrections ($\Omega_\lambda = 1$), parameterized by the values of $Q = \epsilon = \pm 1$ and s_c. It can represent the early (late)-time universe for $\epsilon = +1$ ($\epsilon = -1$) and $\gamma > 1$.

The singular points $Q_{12}^{\pm}(s^*)$ corresponds to a scalar field-dark radiation scaling solution ($\Omega_U \sim \Omega_\phi$), with an effective equation of state parameter of the total matter $w_{eff} = \frac{1}{3}$ corresponding to radiation. These critical points corresponds to a relevant late-time attractor for $\frac{4}{3} < \gamma \leq 2, s^* > 2, f'(s^*) > 0$.

For all the critical point listed above the isotropization has been achieved. The existence of such late-time isotropic solutions (e.g., $Q_4^+(s^*), Q_5^+(s^*), Q_9^+$, and Q_{10}^+), than can attract an initially anisotropic universe, are of significant cosmological interest and have been obtained and discussed in the literature [84, 85]. The fact that an isotropic solution is accompanied by acceleration, makes it a very good candidate for the description of the observable universe.

On the other hand, there are solutions that are anisotropic. For example, the solution $Q_6^{\pm}$, and $Q_7^{\pm}$ represent shear-dominated solution which are always decelerating. Q_6^+ cannot represent accurately the late-time universe, but it has a large probability to represent the early-time universe. Q_6^- is of saddle type. For $Q_7^{\pm}$ we have similar results as for $Q_6^{\pm}$.

The circles of critical points $Q_8^{\pm}$ correspond to scalar field-anisotropic scaling solutions ($\Omega_\phi \sim \Sigma$). Both solutions represents transient states in the evolution of the universe. When $x \to 0$ the anisotropic term in the Friedmann equation (8) dominates the cosmological dynamics ($\Sigma = \pm 1$). In this limit the corresponding cosmological are in a vicinity of the initial singularity.

Summarizing, the possible late time attractors are:

- $Q_4^+(s^*)$ is stable for either $0 < \gamma < \frac{4}{3}, s^* < -\sqrt{3\gamma}, f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}, s^* > \sqrt{3\gamma}, f'(s^*) > 0$;
- $Q_5^+(s^*)$ is stable for either $0 < \gamma \leq \frac{4}{3}, -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0$ or $\frac{4}{3} < \gamma \leq 2, -2 < s^* < 0, f'(s^*) < 0$ or $0 < \gamma \leq \frac{4}{3}, 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0$ or $\frac{4}{3} < \gamma \leq 2, 0 < s^* < 2, f'(s^*) > 0$;
- Q_9^+ and the line Q_{10}^+ are stable for $f(0) \geq 0$. However, if we include the variable $y = \frac{V}{3H^2}$, in the analysis, the solutions associated to the line of fixed points Q_{10}^+ are unstable to perturbations along the y-direction.
- Q_{11}^- is stable for $\gamma > 1$.
- $Q_{12}^+(s^*)$ is stable for $\frac{4}{3} < \gamma \leq 2, s^* > 2, f'(s^*) > 0$;

whereas, the possible past attractors are:

- $Q_4^-(s^*)$ is a source for either $0 < \gamma < \frac{4}{3}, s^* < -\sqrt{3\gamma}, f'(s^*) < 0$ or $0 < \gamma < \frac{4}{3}, s^* > \sqrt{3\gamma}, f'(s^*) > 0$;
- $Q_5^-(s^*)$ is a source for either $0 < \gamma \leq \frac{4}{3}, -\sqrt{3\gamma} < s^* < 0, f'(s^*) < 0$ or $\frac{4}{3} < \gamma \leq 2, -2 < s^* < 0, f'(s^*) < 0$ or $0 < \gamma \leq \frac{4}{3}, 0 < s^* < \sqrt{3\gamma}, f'(s^*) > 0$ or $\frac{4}{3} < \gamma \leq 2, 0 < s^* < 2, f'(s^*) > 0$;
- Q_9^- and the line Q_{10}^- are unstable for $f(0) \geq 0$.
- Q_{11}^+ is unstable for $\gamma > 1$.
- $Q_{12}^-(s^*)$ is source for $\frac{4}{3} < \gamma \leq 2, s^* > 2, f'(s^*) > 0$;

Additionally, our system admits a large class of static solutions that are typically of saddle type. This kind of solutions are important at intermediated stages in the evolution of the universe, since they allow to the transition from expanding to contracting models and vice versa. In fact, the more interesting result for the scenario at hand is that still a large probability for the transition from contracting to expanding universes, since for the same values of the parameters we have a contracting local sources and an expanding local attractors. The possibility of re-collapse in Bianchi I brane worlds has been investigated for example in [44], but in this case for a nonstandard effective equation of state of a brane matter $p = (\gamma - 1)\rho$ with $2 < \gamma < 4$. In [36] the authors used an argument similar to Wald's no-hair theorem [88] for global anisotropy in the brane world scenarios. There were derived a set of sufficient conditions which must be satisfied by the brane matter and bulk metric so that a homogeneous and anisotropic brane asymptotically evolves to a de Sitter spacetime in the presence of a positive cosmological constant on the brane. Following that reference, the isotropy is reached for any initial condition corresponding to $\mathcal{U} \geq 0$ (we have obtained the same result in [24] for a quintessence field with positive potential trapped on the brane by without using Wald's arguments). When $\mathcal{U} < 0$ the brane may not isotropize and can instead collapse. The existence of re-collapsing solutions have been also discussed in [19, 21] for the case of an scalar field with exponential potential on the brane. In the present paper we go a step forward by considering potentials beyond the exponential one.

Finally, a very interesting feature of our scenario is that it allows for bouncing and turnaround, which leads to cyclic solutions [89–93]. In fact, using a heuristic reasoning we have obtained several conditions for the existence of bounces and turnaround of cosmological solutions for a massless scalar field and a scalar field with a potential asymptotically constant. For the analysis of more general potentials we submit the reader to the reference [11] where the authors discuss conditions for bounces and they provide details of chaotic behavior in the isotropic brane model containing an scalar field for a negative dark radiation $\mathcal{U} < 0$. Similarly, in [19] the results correspond to a perfect fluid with equation of state $p = (\gamma - 1)\rho$ whereas in [98] a scalar field on the brane is considered. Bouncing solutions are found to exist both in FRW R^n-gravity [94], as well as in the Bianchi I and Bianchi III R^n-gravity [85] (see also [95]) and in Kantowsky-Sachs R^n-gravity [86]. More interesting, these features can be alternatively obtained in the brane cosmology context [96–100].

The complete bouncing and cyclic analysis in our framework (that goes beyond the heuristic results presented by us in section III), as well as the corresponding perturbation investigation, will not be considered in the present work and it is left for a future project. Strictly speaking, to complete the bounce and cyclic analysis we have to additionally examine the second Friedman equation, that is the equation for $\dot{H}$, since one could have the very improbable case of H from positive to become exactly zero and then positive again, without obtaining negative values at all. In this case we acquire $H = 0$ and $\dot{H} = 0$ simultaneously, that is a universe that stops and starts expanding again without a turnaround.

VI. EXAMPLES OF PHYSICAL POTENTIALS IN THE COSMOLOGICAL LITERATURE

In this section we want to discuss two physical potentials: the cosh-like potential and the inverted sinh-like potential that have been widely investigated in the literature.

The cosh-like potential has been widely studied for example in [48, 60–66]. In [66], and independently by [60], was used a potential of the form $V(\phi) = V_0 [\cosh(\xi\phi) - 1]$ to explain the core density problem for disc galaxy halos in the ΛCDM model (see also [48] and [64], [65]). The asymptotic properties of a cosmological model with a scalar field with cosh-like potential have been investigate in general relativistic framework by [46] and for RS2-FRW branes in [22].

The inverted sinh-like potential has been widely investigated in the literature for example in [48, 61, 63, 67]. The asymptotic properties of a cosmological model with a scalar field with such a potential have been investigated in the context of FRW brane in [22].

A. Potential $V(\phi) = V_0 [\cosh(\xi\phi) - 1]$

Now let us consider the potential

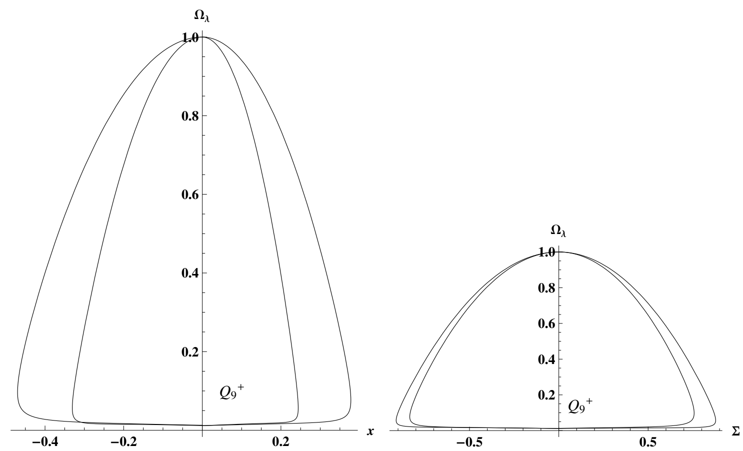
$$V(\phi) = V_0 [\cosh(\xi\phi) - 1] \tag{37}$$

The f-diviser corresponding to this potential is given by

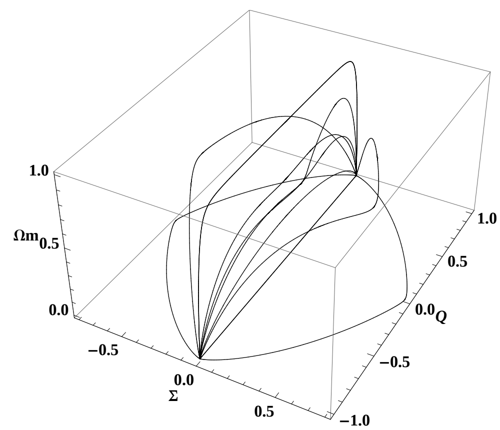
$$f(s) = -\frac{1}{2}(s^2 - \xi^2) \tag{38}$$

which has its zero is

$$s^* = \pm\xi \quad f'(s^*) = \mp\xi \tag{39}$$



(a)



(b)

FIG. 1: (a) Projection of some orbits of the system (21)-(26) for the potential $V(\phi) = V_0\left[\cosh(\xi\phi) - 1\right]$ with $\xi = 1/2$ in the planes $x - \Omega_\lambda$ and $\Omega_\sigma - \Omega_\lambda$ for $\gamma = 1$. This numerical elaboration shows that Q_9^+ is the local attractor.
(b) Projection of some orbits in the space Σ, Q, Ω_m. We choose the initial states near static solutions. This numerical elaboration shows the transition from the contracting de Sitter solution Q_9^- (past attractor) to the expanding de Sitter solution Q_9^+ (future attractor).

The only possible expanding late time attractor of the model with potential (37) is the Sitter attractor Q_9^+

In figure 1 are showed some numerical integrations for the system (21)-(26) for the function (38) with $\xi = 1/2$. This numerical elaboration shows that the expanding de Sitter solution Q_9^+ is the future attractor whereas the contracting de Sitter solution Q_9^- is the past attractor. It is illustrated also the transition from contracting to expanding solutions. In figure 1 (b) we choose initial conditions near the static solutions presented in table IV. This kind of solutions allow for the transition form contracting deceleration to expanding accelerated solutions.

B. Potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$

Let us consider the potential

$$V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi) \tag{40}$$

The f-diviser corresponding to this potential is given by

$$f(s) = \frac{s^2}{\alpha} - \alpha\beta^2 \tag{41}$$

The zeros of this $f(s)$ function are

$$s^* = \pm\alpha\beta \quad f'(s^*) = \pm 2\beta \tag{42}$$

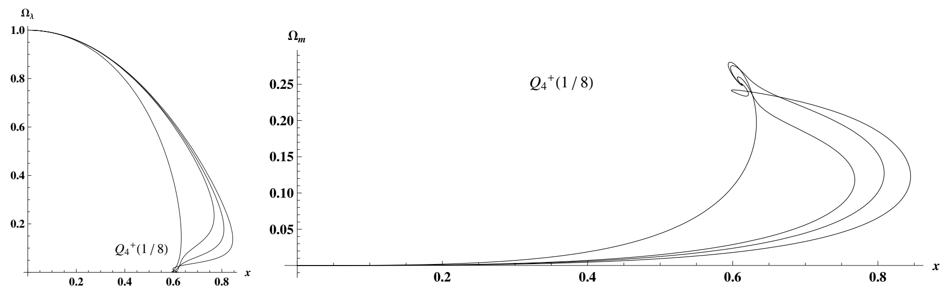
The sufficient conditions for the existence of late-time attractors are fulfilled easily

- The scalar field-matter scaling solution $Q_4^+(\alpha\beta)$ is a late-time attractor provided $0<\gamma<\frac{4}{3}, \beta<0, \alpha>-\frac{\sqrt{3\gamma}}{\beta}$, or $0<\gamma<\frac{4}{3}, \alpha>\frac{\sqrt{3\gamma}}{\beta}, \beta>0$.
- The scalar field-matter scaling solution $Q_4^+(-\alpha\beta)$ is a late-time attractor provided $0<\gamma<\frac{4}{3}, \alpha<\frac{\sqrt{3\gamma}}{\beta}, \beta>0$ or $0<\gamma<\frac{4}{3}, \alpha>-\frac{\sqrt{3\gamma}}{\beta}, \beta<0$. See figure 2.
- The solution dominated by scalar field $Q_5^+(\alpha\beta)$ is a late-time attractor provided $0<\gamma\leq\frac{4}{3}, \beta<0, 0<\alpha<-\frac{\sqrt{3\gamma}}{\beta}$ or $\frac{4}{3}<\gamma\leq 2, \beta<0, 0<\alpha<-\frac{2}{\beta}$ or $0<\gamma\leq\frac{4}{3}, 0<\alpha<\frac{\sqrt{3\gamma}}{\beta}, \beta>0$ or $\frac{4}{3}<\gamma\leq 2, 0<\alpha<\frac{2}{\beta}, \beta>0$;
- The solution dominated by scalar field $Q_5^+(-\alpha\beta)$ is a late-time attractor provided $0<\gamma\leq\frac{4}{3}, \beta>0, 0<\alpha<\frac{\sqrt{3\gamma}}{\beta}$ or $\frac{4}{3}<\gamma\leq 2, \beta>0, 0<\alpha<\frac{2}{\beta}$ or $0<\gamma\leq\frac{4}{3}, 0<\alpha<-\frac{\sqrt{3\gamma}}{\beta}, \beta<0$ or $\frac{4}{3}<\gamma\leq 2, 0<\alpha<-\frac{2}{\beta}, \beta<0$; see figures 3.
- The de Sitter solution Q_9^+ is the late time attractor provided $f(0)=-\alpha\beta^2>0$. See figure 4.
- Q_{11}^- is stable for $\gamma>1$.
- The scalar field- dark radiation scaling solution $Q_{12}^+(\alpha\beta)$ is a late-time attractor for $\frac{4}{3}<\gamma\leq 2, \alpha\beta>2, \beta>0$.
- The scalar field- dark radiation scaling solution $Q_{12}^+(-\alpha\beta)$ is a late-time attractor for $\frac{4}{3}<\gamma\leq 2, \alpha\beta<-2, \beta<0$.

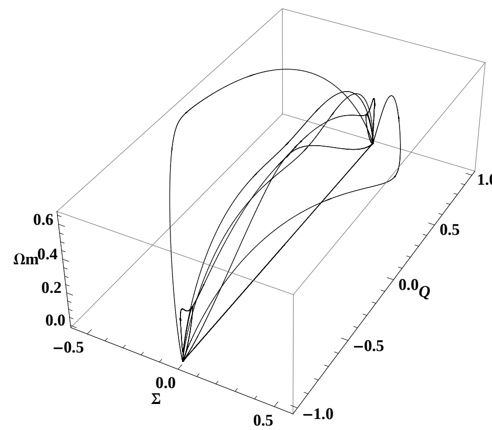
The past attractors of the system are as follows.

- The scalar field-matter scaling solution $Q_4^-(\alpha\beta)$ is the past attractor provided $0<\gamma<\frac{4}{3}, \beta<0, \alpha>-\frac{\sqrt{3\gamma}}{\beta}$, or $0<\gamma<\frac{4}{3}, \alpha>\frac{\sqrt{3\gamma}}{\beta}, \beta>0$.
- The scalar field-matter scaling solution $Q_4^-(-\alpha\beta)$ is the past attractor provided $0<\gamma<\frac{4}{3}, \alpha<\frac{\sqrt{3\gamma}}{\beta}, \beta>0$ or $0<\gamma<\frac{4}{3}, \alpha>-\frac{\sqrt{3\gamma}}{\beta}, \beta<0$. See figure 2 (b).
- The solution dominated by scalar field $Q_5^-(\alpha\beta)$ is the past attractor provided $0<\gamma\leq\frac{4}{3}, \beta<0, 0<\alpha<-\frac{\sqrt{3\gamma}}{\beta}$ or $\frac{4}{3}<\gamma\leq 2, \beta<0, 0<\alpha<-\frac{2}{\beta}$ or $0<\gamma\leq\frac{4}{3}, 0<\alpha<\frac{\sqrt{3\gamma}}{\beta}, \beta>0$ or $\frac{4}{3}<\gamma\leq 2, 0<\alpha<\frac{2}{\beta}, \beta>0$;
- The solution dominated by scalar field $Q_5^-(-\alpha\beta)$ is the past attractor provided $0<\gamma\leq\frac{4}{3}, \beta>0, 0<\alpha<\frac{\sqrt{3\gamma}}{\beta}$ or $\frac{4}{3}<\gamma\leq 2, \beta>0, 0<\alpha<\frac{2}{\beta}$ or $0<\gamma\leq\frac{4}{3}, 0<\alpha<-\frac{\sqrt{3\gamma}}{\beta}, \beta<0$ or $\frac{4}{3}<\gamma\leq 2, 0<\alpha<-\frac{2}{\beta}, \beta<0$; see the figure 3 (b).
- The de Sitter solution Q_9^- is the past attractor provided $f(0)=-\alpha\beta^2>0$. See figure 4 (b)
- Q_{11}^+ is unstable for $\gamma>1$.
- The scalar field- dark radiation scaling solution $Q_{12}^-(\alpha\beta)$ is a past attractor for $\frac{4}{3}<\gamma\leq 2, \alpha\beta>2, \beta>0$.
- The scalar field- dark radiation scaling solution $Q_{12}^-(-\alpha\beta)$ is the past attractor for $\frac{4}{3}<\gamma\leq 2, \alpha\beta<-2, \beta<0$.

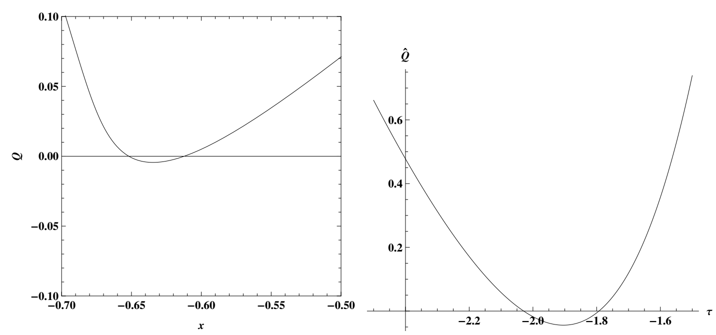
In figure 2 we present some orbits in the phase space of (21)-(26) for the potential $V(\phi)=V_0\sinh^{-\alpha}(\beta\phi)$ with $\alpha=1/2, \beta=4$ and $\gamma=1$. In figure 2 (a) we set $Q=+1$ for the numerical simulation. This numerical elaboration reveals that $Q_4^+(1/8)$ is the future attractor. In figure 2 (b) we present a projection of some orbits in the subspace (Σ, Q, Ω_m). We choose the initial states near static solutions. This numerical elaboration illustrates the transition for the contracting scalar-field matter scaling solution $Q_4^-(1/8)$ (past attractor) to the expanding one $Q_4^+(1/8)$ (future attractor). As can be seen from the figures, the static solutions play an important role in the transition form contracting to expanding solutions and vice versa. In figure 2 (c) we show the transition from expansion to contraction and then



(a)

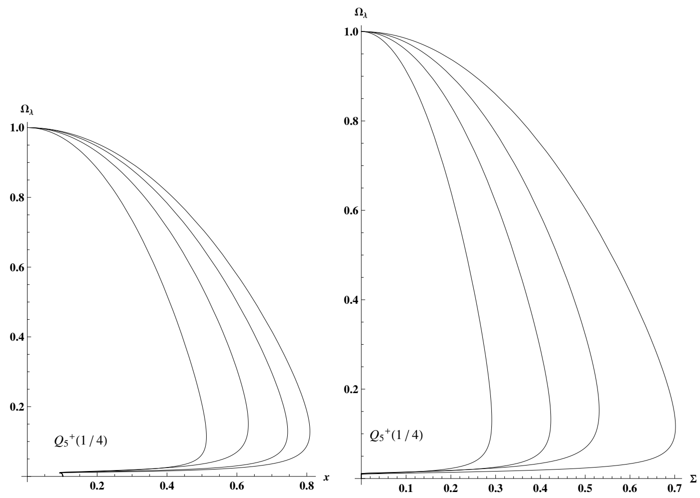


(b)

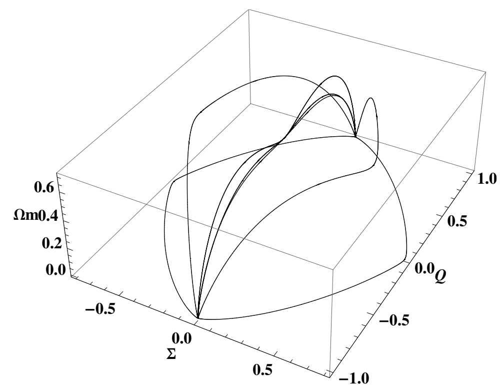


(c)

FIG. 2: Some orbits in the phase space of (21)-(26) for the potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ with $\alpha = 1/2, \beta = 4$ and $\gamma = 1$. (a) Projection in the planes $x - \Omega_\lambda$ and $x - \Omega_m$. We set $Q = +1$ in the numerical simulation. This numerical elaboration shows that $Q_4^+(1/8)$ is the future attractor.
(b) Projection in the subspace (Σ, Q, Ω_m). We choose the initial states near static solutions. This numerical elaboration illustrate the transition for the contracting scalar-field matter scaling solution $Q_4^-(1/8)$ (past attractor) to the expanding one $Q_4^+(1/8)$ (future attractor).
(c) Transition from expansion ($Q > 0$) to contraction ($Q < 0$) and then back to expansion ($Q > 0$), that is a cosmological turnaround and a bounce. The left panel corresponds to the projection of an orbit in the plane x-Q, while the right panel shows $\hat{Q} \equiv 10Q$ vs τ, in order to illustrate that Q change its sign twice during the evolution. We choose the initial condition $Q(0) = 0.9, x(0) = 0.6, \Omega_m(0) = 0.25, \Omega_\lambda(0) = 0, \Sigma(0) = 0, s(0) = 2.0$.



(a)



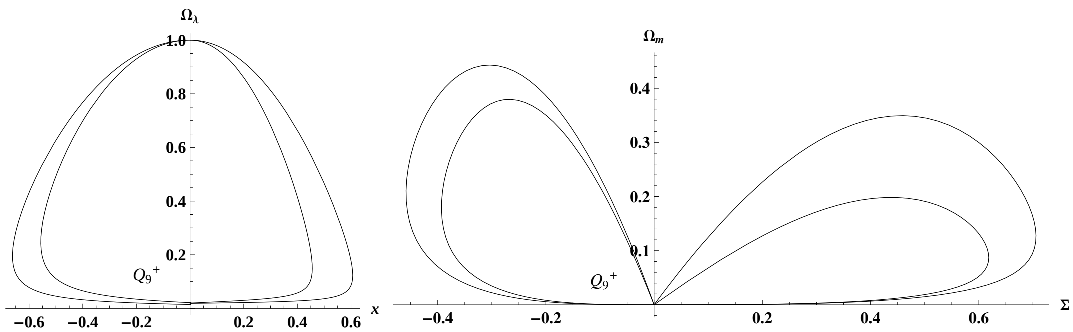
(b)

FIG. 3: Some orbits of the phase space of (21)-(26) for the potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ with $\alpha = \beta = 1/2$ and $\gamma = 1$. (a) Projection in the planes $x - \Omega_\lambda$ and $\Omega_\sigma - \Omega_\lambda$ (We set $Q = +1$ in the numerical simulation). (b) Some orbits in the projection (Σ, Q, Ω_m). We choose initial states near the static solutions. This numerical elaboration shows the transition from the contracting scalar field dominated solution $Q_5^-(1/4)$ to the expanding one $Q_5^+(1/4)$.

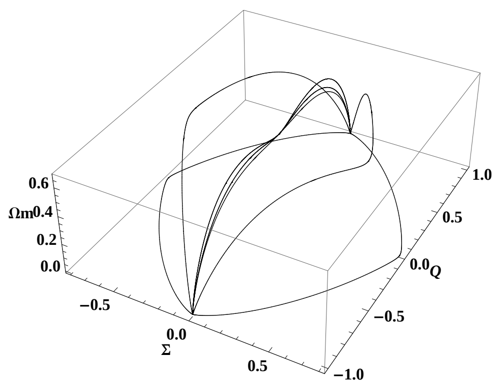
back to expansion, that is a cosmological turnaround and a cosmological bounce, for the solutions of (21)-(26), for the potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ with $\alpha = 1/2, \beta = 4$ and $\gamma = 1$. The left panel corresponds to the projection of an orbit in the plane x-Q, while the right panel shows $\hat{Q} \equiv 10Q$ vs τ in order to illustrate that Q change its sign twice during the evolution. We choose the initial condition $Q(0) = 0.9, x(0) = 0.6, \Omega_m(0) = 0.25, \Omega_\lambda(0) = 0, \Sigma(0) = 0, s(0) = 2.0$.

In figure 3 are presented some orbits in the phase space of (21)-(26) for the potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ with $\alpha = \beta = 1/2$ and $\gamma = 1$. In figure 3 (a) we set $Q = +1$ in the numerical simulation and we present the projection in the planes $x - \Omega_\lambda$ and $\Omega_\sigma - \Omega_\lambda$. In this case the future attractor is the isotropic solution dominated by scalar field $Q_5^+(1/4)$, also this solution corresponds to an accelerated expansion rate since $s^* = 1/4 < \sqrt{2}$. In figure 3 (b) are presented some orbits in the projection (Σ, Q, Ω_m). This numerical elaboration shows the transition from the contracting scalar field dominated solution $Q_5^-(1/4)$ to the expanding one $Q_5^+(1/4)$. We choose the initial states near the static solutions.

In figures 4 are showed some numerical integrations for the system (21)-(26) for the function (41) with $\alpha = -1/2, \beta = -1/4$. In this case the local attractor is the solution dominated by the potential energy of the scalar field Q_9^+ (Sitter attractor). The past attractor is Q_9^-. This numerical elaboration illustrate the transition from contracting to expanding solutions and vice versa that is not allowed for Bianchi I branes with positive Dark Radiation term ($\mathcal{U} > 0$).



(a)



(b)

FIG. 4: Some orbits of the phase space of (21)-(26) for the potential $V(\phi) = V_0 \sinh^{-\alpha}(\beta\phi)$ with $\alpha = -1/2, \beta = -1/4$ and $\gamma = 1$. (a) Projection in the planes $x - \Omega_\lambda$ and $\Sigma - \Omega_m$. We set $Q = +1$ in the numerical simulation.
(b) Projection in the subspace (Σ, Q, Ω_m), $Q \neq +1$. We choose the initial states near static solutions. This numerical elaboration illustrate the transition for the contracting de Sitter solution Q_9^- (past attractor) to the expanding de Sitter solution Q_9^+ (future attractor).

VII. CONCLUSIONS

In this paper we have presented a full phase space analysis of a model consisting of a quintessence field with arbitrary potential and a perfect fluid trapped in a Randall-Sundrum's Braneworld of type 2. We have considered a homogeneous but anisotropic Bianchi I (BI) brane geometry. Moreover, we have also included the effect of the projection of the five-dimensional Weyl tensor onto the three-brane in the form of a negative Dark Radiation term.

We have discussed a general method for the treatment of the potential, called "Method of f-devisers" that allows investigating arbitrary potentials in a phase space. This method is a modification of the method that has been applied to isotropic (FRW) scenarios, and that has been generalized to several cosmological contexts. The "Method of f-devisers" has the significant advantage, that one can first perform the analysis for arbitrary potentials and then just substitute the desired forms, instead of repeating the whole procedure for every distinct potential. More importantly is that the method does not depend on the cosmological scenario, since it is constructed on the scalar field and its self-interacting potential, and can be generalized to several scalar fields. The disadvantages are that such a generalization is only possible if in the model do not appear functions that contains mixed terms of several of the scalar fields, or if the model contains more than one "arbitrary" function, for example non-minimally coupled scalar field cosmologies. In such a case, the more convenient approach is to consider the scalar field itself as a dynamical variable.

By means of the "Method of f-devisers" as well as some tools from the Theory of Dynamical Systems, we have obtained general conditions which have to be satisfied by the potential (encoded in the mathematical properties of the key $f(s)$-functions) in order to obtain the stability conditions of standard 4D and non-standard 5D de Sitter solutions.

We have presented general conditions under the potential for the stability of standard 4D de Sitter solutions. We proved that the de Sitter solutions with 5D corrections are stable against the perturbations introduced here, but are unstable against small perturbations of $V/(3H^2)$. This fact does not conflict with our previous results for BI branes

with positive dark radiation term. Also, we presented the stability conditions for both scalar field-matter scaling solutions, for scalar field-dark radiation scaling solutions and scalar field-dominated solutions. The shear-dominated solutions are always unstable (contracting shear-dominated solutions are of saddle type).

A crucial difference with respect our previous works is that for $\mathcal{U} < 0$ the traditionally ever-expanding models could potentially re-collapse due to the of this dark radiation. Additionally, our system admits a large class of static solutions that are of saddle type. This kind of solutions are important at intermediate stages in the evolution of the universe, since they allows to the transition from contracting to expanding models and vice versa. Thus, a new feature of this scenario is the existence of a bounce and a turnaround, which leads to cyclic behavior. This features are not allowed in Bianchi I branes with positive dark radiation term. This is an important cosmological result in our scenario, that is a consequence of the negativeness of the dark radiation.

Finally, in order to be more transparent, we have illustrated our main results for the specific potentials $V \propto \sinh^{-\alpha}(\beta\phi)$ and $V \propto [\cosh(\xi\phi) - 1]$ which have simple f-devisers.

Acknowledgments

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Appendix A: De Sitter solutions.

To investigate the de Sitter solution Q_9^+, we introduce the local coordinates:

$$\{x, \Omega_\lambda, Q-1, \Omega_m, \Sigma, s\} = \epsilon\left\{\hat{x}, \hat{\Omega}_\lambda, \hat{Q}, \hat{\Omega}_m, \hat{\Omega}_\sigma, \hat{s}\right\} + \mathcal{O}(\epsilon)^2 \tag{A1}$$

where ϵ is a small constant $\epsilon \ll 1$. Then, we obtain the approximated system

$$\begin{aligned}
&\hat{Q}' = -4\hat{Q}, \hat{x}' = \sqrt{\frac{3}{2}}\hat{s} - 3\hat{x}, \hat{\Omega}_\lambda' = 0, \\
&\hat{\Omega}_m' = -3\gamma\hat{\Omega}_m, \hat{\Omega}_\sigma' = -3\hat{\Omega}_\sigma, \hat{s}' = -\sqrt{6}f(0)\hat{x}.
\end{aligned} \tag{A2}$$

The system (A2) admits the exact solution passing by $(\hat{Q}_0, \hat{x}_0, \hat{\Omega}_{\lambda 0}, \hat{\Omega}_{m0}, \hat{\Omega}_{\sigma 0}, \hat{s}_0)$ at $\tau = 0$ given by

$$\begin{aligned}
&\hat{Q}(\tau) = \hat{Q}_0 e^{-4\tau}, \\
&\hat{x}(\tau) = \frac{1}{2}\hat{x}_0 e^{-\frac{1}{2}\tau(\vartheta+3)}\left(e^{\tau\vartheta}+1\right) - \frac{e^{-\frac{1}{2}\tau(\vartheta+3)}\left(e^{\tau\vartheta}-1\right)\left(\sqrt{6}\hat{s}_0(u_c-1)+3\hat{x}_0\right)}{2\vartheta}, \\
&\hat{\Omega}_\lambda(\tau) = \hat{\Omega}_{\lambda 0}, \\
&\hat{\Omega}_m(\tau) = \hat{\Omega}_{m0} e^{-3\gamma\tau}, \hat{\Omega}_\sigma(\tau) = \hat{\Omega}_{\sigma 0} e^{-3\tau}, \\
&\hat{s}(\tau) = \hat{s}_0 e^{-3\tau/2}\cosh(\beta\tau) + \frac{3e^{-3\tau/2}\left(2\hat{s}_0 - \sqrt{6}\hat{x}_0\right)\sinh(\beta\tau)}{4\beta} + \sqrt{\frac{2}{3}}\beta e^{-3\tau/2}\hat{x}_0\sinh(\beta\tau)
\end{aligned} \tag{A3}$$

where $\beta = \frac{1}{2}\sqrt{9 - 12f(0)}$.

For the choice $\beta^2 < \frac{9}{4}$, i.e., $f(0) > 0$, all the perturbations $\hat{Q}, \hat{x}, \hat{\Omega}_m, \hat{\Omega}_\sigma, \hat{s}$ shrink to zero as $\tau \to +\infty$, and $\hat{\Omega}_\lambda$ converges to a constant value. For $\beta = \pm\frac{3}{2}$, i.e., for $f(0) = 0$, $\hat{x} \to \frac{\hat{s}_0}{\sqrt{6}}$ and $\hat{s} \to \hat{s}_0$, and the other perturbations shrink to zero as $\tau \to +\infty$. Combining the above arguments we obtain that for $f(0) \geq 0$, Q_9^+ is stable, but not asymptotically stable. For $\beta^2 > \frac{9}{4}$, i.e., $f(0) < 0$, all the perturbation values, but $\hat{x}$ and $\hat{s}$, which diverges, shrink to zero as $\tau \to +\infty$. Thus, Q_9^+ is a saddle.

For analizing the curve of singular points Q_{10}^{+} we consider an arbitrary value $\Omega_\lambda = u_c$, $0 < u_c < 1$, and introduce the local coordinates

$$\{x, \Omega_\lambda - u_c, Q-1, \Omega_m, \Sigma, s\} = \epsilon\left\{\hat{x}, \hat{\Omega}_\lambda, \hat{Q}, \hat{\Omega}_m, \hat{\Omega}_\sigma, \hat{s}\right\} + \mathcal{O}(\epsilon)^2 \tag{A4}$$

where ϵ is a small constant $\epsilon \ll 1$.

Then, we obtain the approximated system

$$\begin{aligned}
&\hat{Q}' = -4\hat{Q}, \hat{x}' = -3\hat{x} - \sqrt{\frac{3}{2}}\hat{s}(u_c - 1), \hat{\Omega}_\lambda' = -3\gamma u_c \hat{\Omega}_m,\\
&\hat{\Omega}_m' = -3\gamma\hat{\Omega}_m, \hat{\Omega}_\sigma' = -3\hat{\Omega}_\sigma, \hat{s}' = -\sqrt{6}f(0)\hat{x}.
\end{aligned} \tag{A5}$$

The system (A5) admits the exact solution passing by $(\hat{Q}_0, \hat{x}_0, \hat{\Omega}_{\lambda 0}, \hat{\Omega}_{m0}, \hat{\Omega}_{\sigma 0}, \hat{s}_0)$ at $\tau = 0$ given by

$$\begin{aligned}
&\hat{Q}(\tau) = \hat{Q}_0 e^{-4\tau}, \hat{x}(\tau) = \frac{e^{-3\tau/2}\left(\sqrt{6}\hat{s}_0 - 3\hat{x}_0\right)\sinh(\beta\tau)}{2\beta} + e^{-3\tau/2}\hat{x}_0\cosh(\beta\tau),\\
&\hat{\Omega}_\lambda(\tau) = u_c\hat{\Omega}_{m0}\left(e^{-3\gamma\tau} - 1\right) + \hat{\Omega}_{\lambda 0},\\
&\hat{\Omega}_m(\tau) = \hat{\Omega}_{m0}e^{-3\gamma\tau}, \hat{\Omega}_\sigma(\tau) = \hat{\Omega}_{\sigma 0}e^{-3\tau},\\
&\hat{s}(\tau) = \frac{3e^{-\frac{1}{2}\tau(\vartheta+3)}\left(e^{\tau\vartheta} - 1\right)\left(2\hat{s}_0(u_c - 1) + \sqrt{6}\hat{x}_0\right)}{4(u_c - 1)\vartheta} + \frac{1}{2}\hat{s}_0 e^{-\frac{1}{2}\tau(\vartheta+3)}\left(e^{\tau\vartheta} + 1\right) +\\
&-\frac{\hat{x}_0\vartheta e^{-\frac{1}{2}\tau(\vartheta+3)}\left(e^{\tau\vartheta} - 1\right)}{2\sqrt{6}(u_c - 1)}
\end{aligned} \tag{A6}$$

where $\vartheta = \sqrt{12f(0)(u_c - 1) + 9}$. It is easy to see that for $f(0) \geq 0$ the solution is stable, but not asymptotically stable. For $f(0) < 0$ is of saddle type.

It is worthy to mention that the above results could be proved by noticing that Q_{10}^{+} -as 1D set- is normaly hyperbolic since the eigen-direction associated with the null eigenvalue, $(0, 0, 1, 0, 0, 0)^T$, that is, the Ω_λ-axis, is tangent to the set. Thus, the stability issue can be resolved by analyzing the signs of the non-null eigenvalues. However, let us comment that if we include the variable $y = \frac{V}{3H^2}$, in the analysis, the solution is unstable to perturbations along the y-direction.

Appendix B: Asymptotic analysis on the singular surface $\Omega_\lambda + \Sigma^2 = 1$.

In this section we analyze the asymptotic structures of the system (21)-(26) near the critical points at the singular surface $\Omega_\lambda + \Sigma^2 = 1$.

1. Shear-dominated solution.

Observe that the one of the singular points at the surface $\Omega_\lambda + \Sigma^2 = 1$ that represents a shear-dominated solution is Q_6^{+}. To investigate its stability we introduce the local coordinates:

$$\{x, \Omega_\lambda, Q-1, \Omega_m, \Sigma+1, s-s_c\} = \epsilon\left\{\hat{x}, \hat{\Omega}_\lambda, \hat{Q}, \hat{\Omega}_m, \hat{\Omega}_\sigma, \hat{s}\right\} + \mathcal{O}(\epsilon)^2 \tag{B1}$$

where ϵ is a constant satisfying $\epsilon << 1$ and s_c is an arbitrary real value for s. Taylor expanding the resulting system and truncating the second order terms we obtain the approximated system

$$\begin{aligned}
&\hat{Q}' = 2\hat{Q}, \hat{x}' = -\sqrt{\frac{3}{2}}s_c\left(-2\hat{\Omega}_\sigma + \hat{\Omega}_\lambda + \hat{\Omega}_m\right),\\
&\hat{\Omega}_\lambda' = 6\hat{\Omega}_\lambda\left(\frac{\gamma\hat{\Omega}_m}{\hat{\Omega}_\lambda - 2\hat{\Omega}_\sigma} + 1\right), \hat{\Omega}_m' = 3(2-\gamma)\hat{\Omega}_m,\\
&\hat{\Omega}_\sigma' = \frac{3\gamma\hat{\Omega}_\lambda\hat{\Omega}_m}{\hat{\Omega}_\lambda - 2\hat{\Omega}_\sigma} + \frac{3}{2}(4\hat{\Omega}_\sigma - \gamma\hat{\Omega}_m), \hat{s}' = -\sqrt{6}\hat{x}f(s_c).
\end{aligned} \tag{B2}$$

Let us define the rates:

$$\{r_m, r_\sigma\} = \frac{1}{\hat{\Omega}_\lambda}\{\hat{\Omega}_m, \hat{\Omega}_\sigma\} \tag{B3}$$

Let us assume $\gamma > 0$. With the above assumptions we can define the new time variable $N = \gamma \ln a$ which preserves the time arrow. Then we deduce the differential equations

$$r_m' = \frac{6r_m^2}{2r_\sigma - 1} - 3r_m \tag{B4}$$

and

$$r_\lambda' = \frac{3r_m}{2} \tag{B5}$$

where the comma denotes derivatives with respect to N.

The system admits the general solution:

$$r_m = \frac{1}{-2c_1 \cosh(3N - c_2) - 2c_1}, r_\sigma = \frac{2c_1 - 1 + \tanh\left(\frac{1}{2}(3N - c_2)\right)}{4c_1} \tag{B6}$$

or

$$r_m = \frac{1}{-2c_1 \cosh(3N - c_2) - 2c_1}, r_\sigma = \frac{2c_1 - 1 - \tanh\left(\frac{1}{2}(3N - c_2)\right)}{4c_1} \tag{B7}$$

where $c_1 \neq 0$ and c_2 are arbitrary constants. In the general case we have that $r_m \to 0$ and r_σ tends to a constant as $N \to -\infty$.

Thus, as $\tau \to -\infty$ the system (B2) has the asymptotic structure:

$$\begin{aligned} &\hat{Q}' = 2\hat{Q}, \hat{x}' = -\sqrt{\frac{3}{2}} s_c(\hat{\Omega}_\lambda - 2\hat{\Omega}_\sigma), \hat{\Omega}_\lambda' = 6\hat{\Omega}_\lambda, \\ &\hat{\Omega}_m' = 3(2-\gamma)\hat{\Omega}_m, \hat{\Omega}_\sigma' = 6\hat{\Omega}_\sigma, \hat{s}' = -\sqrt{6}\hat{x} f(s_c) \end{aligned} \tag{B8}$$

The system (B8) admits the exact solution passing by $(Q_0, \hat{x}_0, \hat{\Omega}_{\lambda 0}, \hat{\Omega}_{m0}, \hat{\Omega}_{\sigma 0}, \hat{s}_0)$ at $\tau = 0$ given by

$$\begin{aligned} &Q(\tau) = \hat{Q}_0 e^{2\tau}, \hat{x}(\tau) = \hat{x}_0 - \frac{s_c\left(e^{6\tau} - 1\right)(\hat{\Omega}_{\lambda 0} - 2\hat{\Omega}_{\sigma 0})}{2\sqrt{6}}, \hat{\Omega}_\lambda(\tau) = \hat{\Omega}_{\lambda 0} e^{6\tau}, \\ &\hat{\Omega}_m(\tau) = \hat{\Omega}_{m0} e^{3(2-\gamma)\tau}, \hat{\Omega}_\sigma(\tau) = \hat{\Omega}_{\sigma 0} e^{6\tau}, \\ &\hat{s}(\tau) = \frac{1}{12} s_c\left(-6\tau + e^{6\tau} - 1\right) f(s_c)(\hat{\Omega}_{\lambda 0} - 2\hat{\Omega}_{\sigma 0}) - \sqrt{6}\tau \hat{x}_0 f(s_c) + \hat{s}_0. \end{aligned} \tag{B9}$$

Thus, the energy density perturbations goes to zero, and s diverge as $\tau \to -\infty$. For that reason, Q_6^+ cannot be the future attractor of the system, thus Q_6^+ has a large probability to be a past attractor. Using the same approach for Q_6^- we obtain that as $\tau \to +\infty$, the energy density perturbations goes to zero, but s diverge. Thus, although Q_6^- have a large probability to be a late time attractor, actually it is of saddle type. For $Q_7^\pm$ we have similar results as for $Q_6^\pm$. Since the procedure is essentially the same as for $Q_6^\pm$, we omit the details.

2. Solution with 5D-corrections.

Another singular points at the surface $\Omega_\lambda + \Sigma^2 = 1$ are the isotropic solutions with $\Omega_\lambda = 1$ (critical points $Q_{11}^\pm$). For the stability analysis of Q_{11}^+ , we introduce the local coordinates:

$$\{x, \Omega_\lambda - 1, Q - 1, \Omega_m, \Sigma, s - s_c\} = \epsilon\left\{\hat{x}, \hat{\Omega}_\lambda, \hat{Q}, \hat{\Omega}_m, \hat{\Omega}_\sigma, \hat{s}\right\} + \mathcal{O}(\epsilon)^2 \tag{B10}$$

where ϵ is a constant satisfying $\epsilon << 1$ and Q_c, s_c are arbitrary real values of Q, s respectively. Then, we obtain the approximated system

$$\begin{aligned} &\hat{Q}' = -2\hat{Q}(3\gamma r + 2), \hat{x}' = \frac{1}{2}\left(-6\hat{x}(\gamma r + 1) - \sqrt{6}s_c(\hat{\Omega}_\lambda + \hat{\Omega}_m)\right), \\ &\hat{\Omega}_\lambda' = -3\gamma r\hat{\Omega}_\lambda, \hat{\Omega}_m' = -3\gamma(2r + 1)\hat{\Omega}_m, \\ &\hat{\Omega}_\sigma' = -3\hat{\Omega}_\sigma(\gamma r + 1), \hat{s}' = -\sqrt{6}\hat{x} f(s_c). \end{aligned} \tag{B11}$$

where we have defined the rate $r=\frac{\hat{\Omega}_m}{\hat{\Omega}_\lambda}$. The evolution equation for r is given by

$$r' = -3\gamma r(r+1). \tag{B12}$$

The equation (B12) has two singular points $r=0$ with eigenvalue $\frac{dr'}{dr}|_{r=0}=-3\gamma$ and $r=-1$ with eigenvalue $\frac{dr'}{dr}|_{r=-1}=3\gamma$. Then, assuming $\gamma>0$ we have that $r\to -1$ as $\tau\to-\infty$. This argument allow us to prove that as as $\tau\to-\infty$, the system (B11) has the asymptotic structure

$$\begin{aligned}&\hat{Q}' = 2(3\gamma-2)\hat{Q}, \hat{x}' = 3\hat{x}(\gamma-1), \hat{\Omega}_\lambda' = 3\gamma\hat{\Omega}_\lambda,\\ &\hat{\Omega}_m' = 3\gamma\hat{\Omega}_m, \hat{\Omega}_\sigma' = 3\hat{\Omega}_\sigma(\gamma-1), \hat{s}' = -\sqrt{6}\hat{x}f(s_c).\end{aligned} \tag{B13}$$

The system (B13) admits the exact solution passing by $(Q_0, \hat{x}_0, \hat{\Omega}_{\lambda 0}, \hat{\Omega}_{m0}, \hat{\Omega}_{\sigma 0}, \hat{s}_0)$ at $\tau=0$ given by

$$\begin{aligned}&\hat{Q}(\tau) = \hat{Q}_0 e^{2(3\gamma-2)\tau}, \hat{x}(\tau) = \hat{x}_0 e^{3(\gamma-1)\tau}, \hat{\Omega}_\lambda(\tau) = \hat{\Omega}_{\lambda 0} e^{3\gamma\tau},\\ &\hat{\Omega}_m(\tau) = \hat{\Omega}_{m0} e^{3\gamma\tau}, \hat{\Omega}_\sigma(\tau) = \hat{\Omega}_{\sigma 0} e^{3(\gamma-1)\tau},\\ &\hat{s}(\tau) = \hat{s}_0 - \frac{\sqrt{\frac{2}{3}}\hat{x}_0 f(s_c)\left(e^{3(\gamma-1)\tau}-1\right)}{(\gamma-1)}\end{aligned} \tag{B14}$$

Hence, for $\gamma>1$, the perturbations $\hat{Q}, \hat{x}, \hat{\Omega}_\lambda, \hat{\Omega}_m, \hat{\Omega}_\sigma$ shrink to zero as $\tau\to-\infty$. In that limit s tends to a finite value. In this case, the solution under investigation has a 4D unstable manifold. The numerical simulations suggest that the singular point at the surface $\Omega_\lambda+\Sigma^2=1$ with $\Omega_\lambda=1$ is a local source. On the other hand, for $\gamma<1$, the solution behave as a saddle point, since the perturbation values $\hat{x}$ and $\hat{\Omega}_\sigma$ do not tend to zero as $\tau\to-\infty$. Using the same approach for Q_{11}^- we obtain that for $\gamma>1$, Q_{11}^- is stable as $\tau\to+\infty$, whereas for $\gamma<1$ it is of saddle type. These results for Q_{11}^+ and for Q_{11}^- are in agreement with the analogous results for m_+ and for m_- in the reference [19].

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